Chapter 9

Sequences, L'Hôpital's Rule, and Improper Integrals

Section 9.1 Sequences (pp. 439–447)

Quick Review 9.1

1. $f(5)=\dfrac{5}{5+3}=\dfrac{5}{8}$

2. $f(-2)=\dfrac{-2}{-2+3}=-2$

3. $-2+(3-1)(1.5)=1$

4. $-7+(5-1)(3)=5$

5. $1.5(2^{4-1})=12$

6. $-2(1.5^{3-1})=-4.5$

7. $\lim\limits_{x\to\infty}\dfrac{5x^3+2x^2}{3x^4+16x^2}=\lim\limits_{x\to\infty}\dfrac{5x^3}{3x^4}=0$

8. Recall from Chapter 2 that $\lim\limits_{t\to 0}\dfrac{\sin t}{t}=1.$ Let $t=3x$. Then $t\to 0$ as $x\to 0$, and

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(3x)}{x}&=\lim_{t\to 0}\frac{\sin t}{\frac{t}{3}}\\&=3\cdot\lim_{t\to 0}\frac{\sin t}{t}\\&=3\cdot 1\\&=3\end{aligned}$$

9. Let $t=\dfrac{1}{x}$. Then $t\to 0$ as $x\to\infty$, and

$$\begin{aligned}\lim_{x\to\infty}\left(x\sin\left(\frac{1}{x}\right)\right)&=\lim_{t\to 0}\left(\frac{1}{t}\cdot\sin(t)\right)\\&=\lim_{t\to 0}\frac{\sin t}{t}\\&=1\end{aligned}$$

10. $$\begin{aligned}\lim_{x\to\infty}\frac{2x^3+x^2}{x+1}&=\lim_{x\to\infty}\frac{2x^3}{x}\\&=\lim_{x\to\infty}2x^2\\&=\infty,\text{ or does not exist.}\end{aligned}$$

Section 9.1 Exercises

1. $\dfrac{1}{2},\dfrac{2}{3},\dfrac{3}{4},\dfrac{4}{5},\dfrac{5}{6},\dfrac{6}{7};\dfrac{50}{51}$

2. $2,\dfrac{5}{2},\dfrac{8}{3},\dfrac{11}{4},\dfrac{14}{5},\dfrac{17}{6};\dfrac{149}{50}$

3. $2,\dfrac{9}{4},\dfrac{64}{27},\dfrac{625}{256},\dfrac{7776}{3125}\approx 2.48832,$ $\dfrac{117649}{46656}\approx 2.521626;\left(\dfrac{51}{50}\right)^{50}\approx 2.691588$

4. $-2, -2, 0, 4, 10, 18; 2350$

5. $3, 1, -1, -3; -11$

6. $-2, -1, 0, 1; 5$

7. $2, 4, 8, 16; 256$

8. $10, 11, 12.1, 13.31; 19.487171=10(1.1)^7$

9. $1, 1, 2, 3; 21$

10. $-3, 2, -1, 1; 2$

11. **(a)** 3

    (b) $a+7d=-2+7(3)=19$

    (c) $a_n=a_{n-1}+3$

    (d) $a_n=-2+(n-1)(3)=3n-5$

12. **(a)** -2

    (b) $a+7d=15+7(-2)=1$

    (c) $a_n=a_{n-1}-2$

    (d) $a_n=15+(n-1)(-2)=-2n+17$

13. **(a)** $\dfrac{1}{2}$

(b) $a+7d=1+7\left(\frac{1}{2}\right)=\frac{9}{2}$

(c) $a_n=a_{n-1}+\frac{1}{2}$

(d) $a_n=1+(n-1)\left(\frac{1}{2}\right)=\frac{(n+1)}{2}$

14. **(a)** 0.1

(b) $a+7d=3+7(0.1)=3.7$

(c) $a_n=a_{n-1}+0.1$

(d) $a_n=3+(n-1)(0.1)=0.1+2.9$

15. **(a)** $\frac{1}{2}$

(b) $8\left(\frac{1}{2}\right)^8=0.03125$

(c) $a_n=\left(\frac{1}{2}\right)a_{n-1}$

(d) $a_n=8\left(\frac{1}{2}\right)^{n-1}=2^{4-n}$

16. **(a)** 1.5

(b) $(1)(1.5)^8\approx 25.6289$

(c) $a_n=(1.5)\,a_{n-1}$

(d) $a_n=(1)(1.5)^{n-1}=(1.5)^{n-1}$

17. **(a)** -3

(b) $(-3)(-3)^8=(-3)^9=-19,683$

(c) $a_n=-3a_{n-1}$

(d) $a_n=(-3)(-3)^{n-1}=(-3)^n$

18. **(a)** -1

(b) $(5)(-1)^8=5$

(c) $a_n=-a_{n-1}$

(d) $a_n=5(-1)^{n-1}$

19. $d=\frac{7-(-2)}{3}=3$

$a_1=-2-3=-5$

$a_n=a_{n-1}+3$ for all $n\geq 2$

20. $d=\frac{-3-5}{4}=-2$

$a_1=5-(-2)(4)=13$

$a_n=13+(n-1)(-2)=-2n+15$

21. $r=\left(\frac{3,010,000}{3010}\right)^{1/3}=10$

$a_1=\frac{3010}{10^3}=3.01$

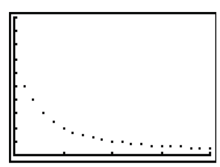
$a_n=3.01(10)^{n-1},\ n\geq 1$

22. $r=\left(\frac{16}{-\frac{1}{2}}\right)^{1/5}=-2$

$a_1=\frac{-\frac{1}{2}}{-2}=\frac{1}{4}$

$a_n=\frac{1}{4}(-2)^{n-1}=(-1)^{n-1}\cdot(2)^{n-3},\ n\geq 1$

23.



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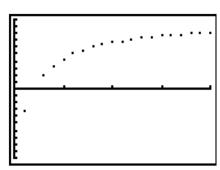

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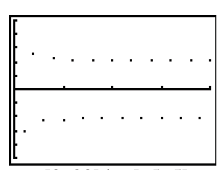

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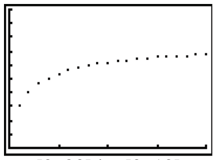
26. 

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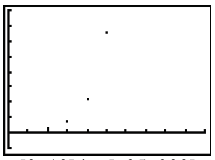
27. 

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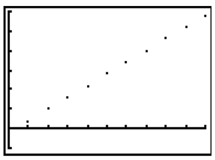
28. 

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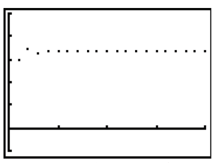
29. 

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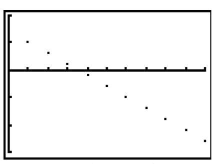
30. 

[0, 10] by [–15, 10]

31. $\lim_{n\to\infty} \frac{3n+1}{n} = \lim_{n\to\infty} (3) + \lim_{n\to\infty} \left(\frac{1}{n}\right) = 3+0 = 3$
converges, 3

32. $\lim_{n\to\infty} \frac{2n}{n+3} = \lim_{n\to\infty} \frac{2n}{n} = 2$
converges, 2

33. $\lim_{n\to\infty} \frac{2n^2-n-1}{5n^2+n+2} = \lim_{n\to\infty} \frac{2n^2}{5n^2} = \frac{2}{5}$
converges, $\frac{2}{5}$

34. $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{n}{n^2} = \lim_{n\to\infty} \frac{1}{n} = 0$
converges, 0

35. $n = 2k,\ \lim_{n\to\infty} (-1)^n \frac{n-1}{n+3} = 1$

$n = 2k-1,\ \lim_{n\to\infty} (-1)^n \frac{n-1}{n+3} = -1$

diverges

36. $n = 2k,\ \lim_{n\to\infty} (-1)^n \frac{n+1}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{n}{n^2} = 0$
$n = 2k - 1,$
$\lim_{n\to\infty} (-1)^n \frac{n+1}{n^2+1} = \lim_{n\to\infty} (-1)^n \frac{n}{n^2} = 0$
converges, 0

37. $\lim_{n\to\infty} (1.1)^n = \infty$
diverges

38. $\lim_{n\to\infty} (0.9)^n = 0$
converges, 0

39. Let $x = \frac{1}{n}$. Then $x \to 0$ as $n \to \infty$, and

$$\lim_{n\to\infty} \left(n \sin\left(\frac{1}{n}\right)\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

converges, 1

40. Diverges; the first terms of the sequence are:
1, 0, −1, 0, 1, 0, −1, 0, 1, 0, −1, 0, ...
The pattern repeats forever, so the sequence does not get close to any one number.

41. $\lim_{n\to\infty} \frac{\sin n}{n} = 0,$ since
$-\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ for $n \ge 1$, and
$\lim_{n\to\infty} \left(-\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0.$

42. $\lim_{n\to\infty} \left(\frac{1}{2^n}\right) = 0,$ since $0 \le \frac{1}{2^n} \le \frac{1}{n}$ for $n \ge 1$,

and $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0.$

43. $\lim_{n\to\infty}\left(\frac{1}{n!}\right)=0$, since $0\le\frac{1}{n!}\le\frac{1}{n}$ $n\ge 1$, and $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0.$

44. $\lim_{n\to\infty}\left(\frac{\sin^2 n}{2^n}\right)=0$, since $0\le\frac{\sin^2 n}{n}\le\frac{1}{n}$ for $n\ge 1$, and $\lim_{n\to\infty}\left(\frac{1}{n}\right)=0.$

45. Graph (b)

46. Graph (c)

47. Table (d)

48. Table (a)

49. False; if the sequence is increasing, the terms will eventually become positive. Consider the sequence with nth term $a_n=-5+2(n-1)$. Here $a=-5$, $a_2=-3$, $a_3=-1$, and $a_4=1$.

50. True; $a_1>0$, $r=\frac{a_2}{a_1}>0$, and

$a_n=a_1\,r^{n-1}>0$ for all $n\ge 2$.

51. C; $d=\frac{5-(-1)}{2}=3$

$a_6=-1+3(5)=14$

52. E; $r=\frac{1.25}{2.5}=\frac{1}{2}$

$a_1=\frac{2.5}{\frac{1}{2}}=5$

53. D; let $x=\frac{3\pi}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\begin{aligned}\lim_{n\to\infty}\left(n\sin\left(\frac{3\pi}{n}\right)\right)&=\lim_{x\to 0}\left(\frac{3\pi}{x}\cdot\sin(x)\right)\\&=3\pi\cdot\lim_{x\to 0}\frac{\sin x}{x}\\&=3\pi\cdot 1\\&=3\pi\end{aligned}$$

54. E; $n=2k$, $\lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=3$

$n=2k-1$, $\lim_{n\to\infty}\left((-1)^n\frac{3n-1}{n+2}\right)=-3$

55. **(a)** Draw a segment from the center of the circle to each vertex of the polygon, forming n isosceles triangles. The vertex angle in each triangle is $\frac{2\pi}{n}$. An altitude to the base divides the isosceles triangle into two right triangles with hypotenuse 1. In each of these triangles, the side opposite the angle of measure $\frac{\pi}{n}$ has length $\sin\left(\frac{\pi}{n}\right)$. It follows that each side of the polygon has length $2\sin\left(\frac{\pi}{n}\right)$ and the total perimeter is $2n\sin\left(\frac{\pi}{n}\right)$.

(b) Let $x=\frac{\pi}{n}$. Then $x\to 0$ as $n\to\infty$, and

$$\begin{aligned}\lim_{n\to\infty}\left(2n\sin\left(\frac{\pi}{n}\right)\right)&=\lim_{x\to 0}\left(2\cdot\frac{\pi}{x}\cdot\sin(x)\right)\\&=2\pi\cdot\lim_{x\to 0}\frac{\sin x}{x}\\&=2\pi\cdot 1\\&=2\pi\end{aligned}$$

56. **(a)** $a_n=a_{n-2}+a_{n-1}$ for $n\ge 3$

$a_1=1;\ a_2=1$

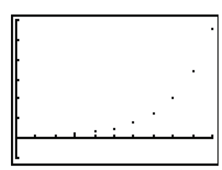
$a_3=a_1+a_2=1+1=2$

$a_4=a_2+a_3=1+2=3$

Continue in this fashion to get all of the first ten terms:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55

(b)



[0, 10] by [–10, 60]

57. $a_n=ar^{n-1}$ implies that

$\log a_n=\log a+(n-1)\log r$. Thus $\{\log a_n\}$ is an arithmetic sequence with first term $\log a$ and common difference $\log r$.

58. $a_n=a+(n-1)d$ implies that

$10^{a_n}=10^{a+(n-1)d}=10^a(10^d)^{n-1}$. Thus $\{10^{a_n}\}$ is a geometric sequence with first term 10^a and common ratio 10^d.

59. Given $\varepsilon > 0$ choose $M = \frac{1}{\varepsilon}$. Then

$$\left|\frac{1}{n} - 0\right| < \varepsilon \text{ if } n > M.$$

Section 9.2 L'Hôpital's Rule (pp. 448–456)

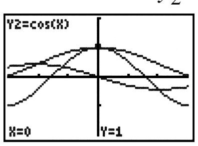
Exploration 1 Exploring L'Hôpital's Rule Graphically

1. $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

2. The two graphs suggest that

$$\lim_{x\to 0} \frac{y_1}{y_2} = \lim_{x\to 0} \frac{y_1'}{y_2'}.$$

3. $y_5 = \frac{x\cos x - \sin x}{x^2}$. The graphs of y_3 and y_5 clearly show that l'Hôpital's Rule does not say that $\lim_{x\to 0} \frac{y_1}{y_2}$ is equal to $\lim_{x\to 0} \left(\frac{y_1}{y_2}\right)'$.



[–3, 3] by [–2, 2]

Quick Review 9.2

1.

x	$\left(1+\frac{0.1}{x}\right)^x$
1	1.1000
10	1.1046
100	1.1051
1000	1.1052
10,000	1.1052
1,000,000	1.1052

As $x \to \infty$, $\left(1+\frac{0.1}{x}\right)^x$ approaches 1.1052.

2.

x	$x^{1/(\ln x)}$
0.1	2.7183
0.01	2.7183
0.001	2.7183
0.0001	2.7183
0.00001	2.7183

As $x \to 0^+$, $x^{1/(\ln x)}$ approaches 2.7183.

3.

x	$\left(1-\frac{1}{x}\right)^x$
– 1	0.5
– 0.1	0.78679
– 0.01	0.95490
– 0.001	0.99312
– 0.0001	0.99908
– 0.00001	0.99988
– 0.000001	0.99999

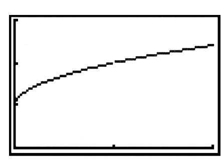
As $x \to 0^-$, $\left(1-\frac{1}{x}\right)^x$ approaches 1.

4.

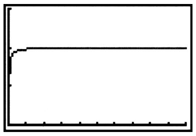
x	$\left(1+\frac{1}{x}\right)^x$
–1.1	13.981
–1.01	105.77
–1.001	1007.9
–1.0001	10010

As $x \to -1^-$, $\left(1+\frac{1}{x}\right)^x$ goes to ∞.

5.

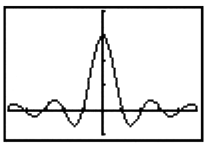

[0, 2] by [0, 3]

As $t \to 1$, $\frac{t-1}{\sqrt{t}-1}$ approaches 2.

6. 

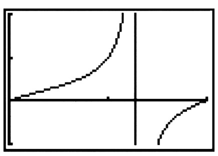
[0, 500] by [0, 3]

As $x \to \infty$, $\dfrac{\sqrt{4x^2+1}}{x+1}$ approaches 2.

7. 

[–5, 5] by [–1, 4]

As $x \to 0$, $\dfrac{\sin 3x}{x}$ approaches 3.

8. 

[0, π] by [–1, 2]

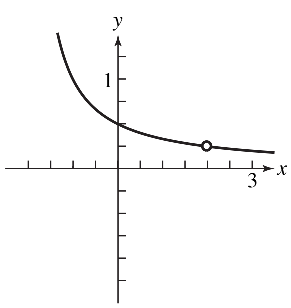
As $\theta \to \dfrac{\pi}{2}$, $\dfrac{\tan\theta}{2+\tan\theta}$ approaches 1.

9. $y = \dfrac{1}{h}\sin h = \dfrac{\sin h}{h}$

10. $y = (1+h)^{1/h}$

Section 9.2 Exercises

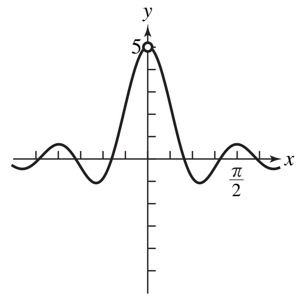
1. $\lim\limits_{x\to 2}\left(\dfrac{x-2}{x^2-4}\right)$ appears to be about $\dfrac{1}{4}$;



By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) = \lim_{x\to 2}\frac{1}{2x} = \frac{1}{4}$$

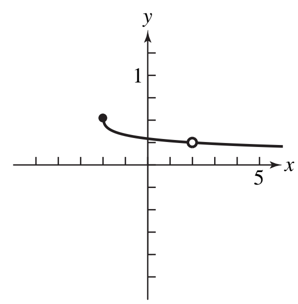
2. $\lim\limits_{x\to 0}\left(\dfrac{\sin(5x)}{x}\right)$ appears to be about 5;



By L'Hôpital's Rule:

$$\lim_{x\to 0}\left(\frac{\sin(5x)}{x}\right) = \lim_{x\to 0}\frac{5\cos(5x)}{1} = 5\cos(0) = 5$$

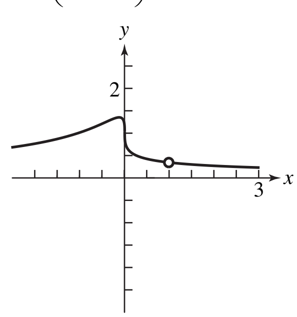
3. $\lim\limits_{x\to 2}\left(\dfrac{\sqrt{2+x}-2}{x-2}\right)$ appears to be about $\dfrac{1}{4}$;



By L'Hôpital's Rule:

$$\lim_{x\to 2}\left(\frac{\sqrt{2+x}-2}{x-2}\right) = \lim_{x\to 2}\left(\frac{\frac{1}{2}(2+x)^{-1/2}}{1}\right)$$
$$= \frac{1}{2}(2+2)^{-1/2}$$
$$= \frac{1}{2\sqrt{4}}$$
$$= \frac{1}{4}$$

4. $\lim\limits_{x\to 1}\left(\dfrac{\sqrt[3]{x}-1}{x-1}\right)$ appears to be about $\dfrac{1}{3}$;



By L'Hôpital's Rule:

$$\lim_{x\to 1}\left(\frac{\sqrt[3]{x}-1}{x-1}\right)=\lim_{x\to 1}\left(\frac{\frac{1}{3}(x)^{-2/3}}{1}\right)$$
$$=\frac{1}{3}(1)^{-2/3}$$
$$=\frac{1}{3}$$

5. $$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)=\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)$$
$$=\lim_{x\to 0}\left(\frac{\cos x}{2}\right)$$
$$=\frac{\cos(0)}{2}$$
$$=\frac{1}{2}$$

6. $$\lim_{\theta\to\pi/2}\left(\frac{1-\sin\theta}{1+\cos(2\theta)}\right)=\lim_{\theta\to\pi/2}\left(\frac{-\cos\theta}{-2\sin(2\theta)}\right)$$
$$=\lim_{\theta\to\pi/2}\left(\frac{\sin\theta}{-4\cos(2\theta)}\right)$$
$$=\frac{\sin\frac{\pi}{2}}{-4\cos\pi}$$
$$=\frac{1}{4}$$

7. $$\lim_{t\to 0}\left(\frac{\cos t-1}{e^t-t-1}\right)=\lim_{t\to 0}\left(\frac{-\sin t}{e^t-1}\right)$$
$$=\lim_{t\to 0}\left(\frac{-\cos t}{e^t}\right)$$
$$=\frac{-\cos(0)}{e^0}$$
$$=-1$$

8. $$\lim_{x\to 2}\left(\frac{x^2-4x+4}{x^3-12x+16}\right)=\lim_{x\to 2}\left(\frac{2x-4}{3x^2-12}\right)$$
$$=\lim_{x\to 2}\left(\frac{2}{6x}\right)$$
$$=\frac{2}{6(2)}$$
$$=\frac{1}{6}$$

9. (a) $$\lim_{x\to 0^-}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^-}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)$$
$$=\frac{4\cos(0)}{2\cos(0)}$$
$$=2$$

(b) $$\lim_{x\to 0^+}\left(\frac{\sin 4x}{\sin 2x}\right)=\lim_{x\to 0^+}\left(\frac{4\cos(4x)}{2\cos(2x)}\right)$$
$$=\frac{4\cos(0)}{2\cos(0)}$$
$$=2$$

10. (a) $$\lim_{x\to 0^-}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{1}\right)=1$$

(b) $$\lim_{x\to 0^+}\left(\frac{\tan x}{x}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{1}\right)=1$$

11. (a) $$\lim_{x\to 0^-}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^-}\left(\frac{\cos x}{3x^2}\right)=\infty$$

(b) $$\lim_{x\to 0^+}\left(\frac{\sin x}{x^3}\right)=\lim_{x\to 0^+}\left(\frac{\cos x}{3x^2}\right)=\infty$$

12. (a) $$\lim_{x\to 0^-}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^-}\left(\frac{\sec^2 x}{2x}\right)=-\infty$$

(b) $$\lim_{x\to 0^+}\left(\frac{\tan x}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{\sec^2 x}{2x}\right)=\infty$$

13. Left:

$$\lim_{x\to\pi^-}\left(\frac{\csc x}{1+\cot x}\right)=\frac{\infty}{-\infty}$$

$$\lim_{x\to\pi^-}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to\pi^-}\left(\frac{\cot x}{\csc x}\right)$$
$$=\lim_{x\to\pi^-}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$$
$$=\lim_{x\to\pi^-}\cos x$$
$$=-1$$

Right:

$$\lim_{x\to\pi^+}\left(\frac{\csc x}{1+\cot x}\right)=\frac{-\infty}{\infty}$$

$$\lim_{x\to\pi^+}\left(\frac{-\csc x\cot x}{-\csc^2 x}\right)=\lim_{x\to\pi^+}\left(\frac{\cot x}{\csc x}\right)$$

$$=\lim_{x\to\pi^+}\frac{\left(\frac{\cos x}{\sin x}\right)}{\left(\frac{1}{\sin x}\right)}$$

$$=\lim_{x\to\pi^+}\cos x$$

$$=-1$$

14. Left:

$$\lim_{x\to\pi/2^-}\left(\frac{1+\sec x}{\tan x}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\pi/2^-}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to\pi/2^-}\left(\frac{\tan x}{\sec x}\right)$$

$$=\lim_{x\to\pi/2^-}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}$$

$$=\lim_{x\to\pi/2^-}\sin x$$

$$=1$$

Right:

$$\lim_{x\to\pi/2^+}\left(\frac{1+\sec x}{\tan x}\right)=\frac{-\infty}{-\infty}$$

$$\lim_{x\to\pi/2^+}\left(\frac{\sec x\tan x}{\sec^2 x}\right)=\lim_{x\to\pi/2^+}\left(\frac{\tan x}{\sec x}\right)$$

$$=\lim_{x\to\pi/2^+}\frac{\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)}$$

$$=\lim_{x\to\pi/2^+}\sin x$$

$$=1$$

15. $$\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{\frac{1}{x+1}}{\frac{1}{x\ln 2}}\right)=\lim_{x\to\infty}\left(\frac{x\ln 2}{x+1}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{\ln 2}{1}\right)=\ln 2$$

16. $$\lim_{x\to\infty}\left(\frac{5x^2-3x}{7x^2+1}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{10x-3}{14x}\right)=\frac{\infty}{\infty}$$

$$\lim_{x\to\infty}\left(\frac{10}{14}\right)=\frac{5}{7}$$

$$Y1=\frac{(5x^2-3x)}{(7x^2+1)}$$

$$\text{limit}=\frac{5}{7}$$

17. $$\lim_{x\to0^+}(x\ln x)=0\cdot\infty$$

$$\lim_{x\to0^+}\left(\frac{\ln x}{1/x}\right)=\lim_{x\to0^+}\left(\frac{1/x}{-1/x^2}\right)=\lim_{x\to0^+}(-x)=0$$

18. $$\lim_{x\to\infty}\left(x\tan\left(\frac{1}{x}\right)\right)=\infty\cdot0$$

Let $h=\frac{1}{x}$. Then $h\to0$ as $x\to\infty$, and

$$\lim_{x\to\infty}x\tan\left(\frac{1}{x}\right)=\lim_{h\to0}\left(\frac{1}{h}\tan(h)\right)$$

$$=\lim_{h\to0}\left(\frac{\tan(h)}{h}\right)$$

$$=\frac{\infty}{\infty}$$

$$\lim_{h\to0}\frac{\sec^2 h}{1}=\sec^2(0)=1$$

19. $$\lim_{x\to0^+}(\csc x-\cot x+\cos x)$$

$$=\lim_{x\to0^+}(\csc x-(\cot x-\cos x))$$

$$=\infty-\infty$$

$$=\lim_{x\to0^+}\frac{1-\cos x+\cos x\sin x}{\sin x}$$

$$=\lim_{x\to0^+}\frac{\sin x-\sin^2 x+\cos^2 x}{\cos x}$$

$$=\frac{0-0+1}{1}$$

$$=1.$$

20. $$\lim_{x\to\infty}(\ln(2x)-\ln(x+1))=\infty-\infty$$

$$\ln(2x)-\ln(x+1)=\ln\left(\frac{2x}{x+1}\right)$$

$$\lim_{x\to\infty}\left(\frac{2x}{x+1}\right)=\lim_{x\to\infty}\frac{2}{1}=2$$

Therefore:

$$\lim_{x\to\infty}(\ln(2x)-\ln(x+1)) = \lim_{x\to\infty}\left(\ln\left(\frac{2x}{x+1}\right)\right)$$
$$= \ln(2)$$

21. $\lim_{x\to 0}(e^x+x)^{1/x} = (1+0)^{\infty} = 1^{\infty}$

$$\ln f(x) = \ln[(e^x+x)^{1/x}] = \frac{\ln(e^x+x)}{x}$$

$$\lim_{x\to 0}\left(\frac{\ln(e^x+x)}{x}\right) = \lim_{x\to 0}\frac{\frac{1}{e^x+x}\cdot(e^x+1)}{1}$$
$$= \lim_{x\to 0}\frac{e^x+1}{e^x+x}$$
$$= 2$$

Therefore:

$$\lim_{x\to 0}(e^x+x)^{1/x} = \lim_{x\to 0} f(x)$$
$$= \lim_{x\to 0} e^{\ln f(x)}$$
$$= e^2$$

22. $\lim_{x\to 1} x^{1/(x-1)} = 1^{\infty}$

$$\ln f(x) = \ln[x^{1/(x-1)}] = \frac{\ln x}{x-1}$$

$$\lim_{x\to 1}\frac{\ln x}{x-1} = \lim_{x\to 1}\frac{\frac{1}{x}}{1} = 1$$

Therefore:

$$\lim_{x\to 1} x^{1/(x-1)} = \lim_{x\to 1} f(x) = \lim_{x\to 1} e^{\ln f(x)} = e^1 = e$$

23. $\lim_{x\to 1}(x^2-2x+1)^{x-1} = (1^2-2(1)+1)^{1-1} = 0^0$

$$\ln f(x) = (x-1)\ln(x^2-2x+1)$$
$$= \frac{\ln(x^2-2x+1)}{1/(x-1)}$$

$$\lim_{x\to 1}\frac{\ln(x^2-2x+1)}{\frac{1}{x-1}} = \lim_{x\to 1}\frac{\frac{2}{x-1}}{-\frac{1}{(x-1)^2}}$$
$$= \lim_{x\to 1}\left(\frac{-2(x-1)^2}{x-1}\right)$$
$$= 0$$

Therefore:

$$\lim_{x\to 1}(x^2-2x+1)^{x-1} = \lim_{x\to 1} f(x)$$
$$= \lim_{x\to 1} e^{\ln f(x)}$$
$$= e^0$$
$$= 1$$

24. $\lim_{x\to 0^+}(\sin x)^x = 0^0$

$$\ln f(x) = \ln[(\sin x)^x] = x\ln(\sin x) = \frac{\ln(\sin x)}{\frac{1}{x}}$$

$$\lim_{x\to 0^+}\frac{\ln(\sin x)}{\frac{1}{x}} = \lim_{x\to 0^+}\left(\frac{\frac{1}{\tan x}}{-\frac{1}{x^2}}\right)$$
$$= \lim_{x\to 0^+}\left(-\frac{x^2}{\tan x}\right)$$
$$= \lim_{x\to 0^+}\left(-\frac{2x}{\sec^2 x}\right)$$
$$= -\frac{0}{1}$$
$$= 0$$

Therefore:

$$\lim_{x\to 0^+}(\sin x)^x = \lim_{x\to 0^+} f(x)$$
$$= \lim_{x\to 0^+} e^{\ln f(x)}$$
$$= e^0$$
$$= 1$$

25. $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = (1+\infty)^0 = \infty^0$

$$\ln f(x) = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to 0^+}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x} = \lim_{x\to 0^+}\left(\frac{\frac{-1}{x(x+1)}}{-\frac{1}{x^2}}\right)$$
$$= \lim_{x\to 0^+}\frac{x^2}{x(x+1)}$$
$$= \lim_{x\to 0^+}\frac{x}{x+1}$$
$$= 0.$$

Therefore:

$$\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0^+} f(x)$$
$$= \lim_{x\to 0^+} e^{\ln f(x)}$$
$$= e^0$$
$$= 1$$

26. $\lim_{x\to\infty} (\ln x)^{1/x} = \infty^0$

$\ln f(x) = \ln[(\ln x)^{1/x}] = \dfrac{\ln(\ln x)}{x}$

$$\begin{aligned}\lim_{x\to\infty} \frac{\ln(\ln x)}{x} &= \lim_{x\to\infty}\left(\frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}\right)\\ &= \lim_{x\to\infty}\left(\frac{1}{x\ln x}\right)\\ &= 0.\end{aligned}$$

Therefore:

$$\begin{aligned}\lim_{x\to\infty} (\ln x)^{1/x} &= \lim_{x\to\infty} f(x)\\ &= \lim_{x\to\infty} e^{\ln f(x)}\\ &= e^0\\ &= 1\end{aligned}$$

27. **(a)**

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.1513	0.2303	0.0345	0.00461	0.00058

Estimate the limit to be 0.

(b) $\lim_{x\to\infty} \dfrac{\ln x^5}{x} = \lim_{x\to\infty} \dfrac{5\ln x}{x} = \lim_{x\to\infty} \dfrac{\frac{5}{x}}{1} = \dfrac{0}{1} = 0$

28. **(a)**

x	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$f(x)$	0.1585	0.1666	0.1667	0.1667	0.1667

Estimate the limit to be $\dfrac{1}{6}$.

(b)
$$\begin{aligned}\lim_{x\to 0^+} \frac{x-\sin x}{x^3} &= \lim_{x\to 0^+} \frac{1-\cos x}{3x^2}\\ &= \lim_{x\to 0^+} \frac{\sin x}{6x}\\ &= \lim_{x\to 0^+} \frac{\cos x}{6}\\ &= \frac{1}{6}\end{aligned}$$

29. Let $f(\theta) = \dfrac{\sin 3\theta}{\sin 4\theta}$.

θ	$\pm 10^0$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$	$\pm 10^{-4}$
$f(\theta)$	–0.1865	0.7589	0.7501	0.7500	0.7500

Estimate the limit to be $=\frac{3}{4}$.

$$\lim_{\theta\to 0}\frac{\sin 3\theta}{\sin 4\theta}=\lim_{\theta\to 0}\frac{3\cos 3\theta}{4\cos 4\theta}=\frac{3}{4}$$

30. Let $f(t)=\frac{1}{\sin t}-\frac{1}{t}=\frac{t-\sin t}{t\sin t}$.

t	$\pm 10^{0}$	$\pm 10^{-1}$	$\pm 10^{-2}$	$\pm 10^{-3}$
$f(t)$	±0.1884	±0.0167	±0.0017	±0.00017

Estimate the limit to be 0.

$$\begin{aligned}\lim_{t\to 0}\left(\frac{1}{\sin t}-\frac{1}{t}\right)&=\lim_{t\to 0}\frac{t-\sin t}{t\sin t}\\&=\lim_{t\to 0}\frac{1-\cos t}{t\cos t+\sin t}\\&=\lim_{t\to 0}\frac{\sin t}{-t\sin t+\cos t+\cos t}\\&=0\end{aligned}$$

31. Let $f(x)=(1+x)^{1/x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	1.271	1.0472	1.0069	1.0009	1.0001

Estimate the limit to be 1.

$$\ln f(x)=\frac{\ln(1+x)}{x}$$

$$\lim_{x\to\infty}\frac{\ln(1+x)}{x}=\lim_{x\to\infty}\frac{\frac{1}{1+x}}{1}=\frac{0}{1}=0$$

$$\lim_{x\to\infty}(1+x)^{1/x}=\lim_{x\to\infty}f(x)=\lim_{x\to\infty}e^{\ln f(x)}=e^{0}=1$$

32. Let $f(x)=\frac{x-2x^2}{3x^2+5x}$.

x	10	10^2	10^3	10^4	10^5
$f(x)$	–0.5429	–0.6525	–0.6652	–0.6665	–0.6667

Estimate the limit to be $-\frac{2}{3}$.

$$\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}=\lim_{x\to\infty}\frac{1-4x}{6x+5}=\lim_{x\to\infty}-\frac{4}{6}=-\frac{2}{3}.$$

33. $\lim_{\theta\to 0}\frac{\sin\theta^2}{\theta}=\lim_{\theta\to 0}\frac{2\theta\cos\theta^2}{1}$
$=(2)(0)\cos(0)^2$
$=0$

34. $\lim_{t\to 1}\frac{t-1}{\ln t-\sin\pi t}=\lim_{t\to 1}\frac{1}{\frac{1}{t}-\pi\cos\pi t}$
$=\frac{1}{1-\pi(-1)}$
$=\frac{1}{\pi+1}$

35. $\lim_{x\to\infty}\frac{\log_2 x}{\log_3(x+3)}=\lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{(x+3)\ln 3}}$
$=\lim_{x\to\infty}\frac{(x+3)\ln 3}{x\ln 2}$
$=\lim_{x\to\infty}\frac{x\ln 3+3\ln 3}{x\ln 2}$
$=\lim_{x\to\infty}\frac{\ln 3}{\ln 2}$
$=\frac{\ln 3}{\ln 2}$

36. $\lim_{y\to 0^+}\frac{\ln(y^2+2y)}{\ln y}=\lim_{y\to 0^+}\frac{\frac{2y+2}{y^2+2y}}{\frac{1}{y}}$
$=\lim_{y\to 0^+}\frac{y(2y+2)}{y^2+2y}$
$=\lim_{y\to 0^+}\frac{(2y^2+2y)}{y^2+2y}$
$=\lim_{y\to 0^+}\frac{4y+2}{2y+2}$
$=\frac{4(0)+2}{2(0)+2}$
$=\frac{2}{2}$
$=1$

37. $\lim_{y\to\pi/2}\left(\frac{\pi}{2}-y\right)\tan y$
$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\sin y}{\cos y}$
$=\lim_{y\to\pi/2}\frac{\left(\frac{\pi}{2}-y\right)\cos y+(-1)\sin y}{-\sin y}$
$=\frac{\left(\frac{\pi}{2}-\frac{\pi}{2}\right)\cos\frac{\pi}{2}+(-1)\sin\frac{\pi}{2}}{-\sin\frac{\pi}{2}}$
$=\frac{(-1)(1)}{-(1)}$
$=1$

38. $\lim_{x\to 0^+}(\ln x-\ln\sin x)=\lim_{x\to 0^+}\ln\frac{x}{\sin x}$

Let $f(x)=\frac{x}{\sin x}$.

$\lim_{x\to 0^+}\frac{x}{\sin x}=\lim_{x\to 0^+}\frac{1}{\cos x}=1$. Therefore,
$\lim_{x\to 0^+}(\ln x-\ln\sin x)=\lim_{x\to 0^+}\ln f(x)=\ln 1=0$

39. This problem does not require L'Hôpital's Rule.

$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sqrt{x}}\right)=\lim_{x\to 0^+}\frac{1-\sqrt{x}}{x}=\infty$

40. The limit leads to the indeterminate form ∞^0.

Let $f(x)=\left(\frac{1}{x^2}\right)^x$.

$\ln\left(\frac{1}{x^2}\right)^x=x\ln\left(\frac{1}{x^2}\right)=\frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}$

$\lim_{x\to 0}\frac{\ln\left(\frac{1}{x^2}\right)}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{\frac{-2}{x^3}}{\frac{1}{x^2}}}{-\frac{1}{x^2}}=\lim_{x\to 0}2x=0$

$\lim_{x\to 0}\left(\frac{1}{x^2}\right)^x=\lim_{x\to 0}e^{\ln f(x)}=e^0=1$

41. $\lim_{x\to\pm\infty}\frac{3x-5}{2x^2-x+2}=\lim_{x\to\pm\infty}\frac{3}{4x-1}=0$

42. $\lim_{x\to 0}\frac{\sin 7x}{\tan 11x}=\lim_{x\to 0}\frac{7\cos 7x}{11\sec^2 11x}=\frac{7}{11}$

43. The limit leads to the indeterminate form ∞^0.

Let $f(x) = (1+2x)^{1/(2\ln x)}$.

$$\ln(1+2x)^{1/(2\ln x)} = \frac{\ln(1+2x)}{2\ln x}$$

$$\lim_{x\to\infty} \frac{\ln(1+2x)}{2\ln x} = \lim_{x\to\infty} \frac{\frac{2}{1+2x}}{\frac{2}{x}} = \lim_{x\to\infty} \frac{x}{1+2x} = \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

$$\lim_{x\to\infty} (1+2x)^{1/(2\ln x)} = \lim_{x\to\infty} e^{\ln f(x)} = e^{1/2} = \sqrt{e}$$

44. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\cos x)^{\cos x}$.

$$\ln(\cos x)^{\cos x} = (\cos x)\ln(\cos x) = \frac{\ln(\cos x)}{\sec x}$$

$$\lim_{x\to\pi/2^-} \frac{\ln(\cos x)}{\sec x} = \lim_{x\to\pi/2^-} \frac{\frac{-\sin x}{\cos x}}{\sec x\tan x} = \lim_{x\to\pi/2^-} \frac{-\tan x}{\sec x\tan x} = \lim_{x\to\pi/2^-} -\cos x = 0$$

$$\lim_{x\to\frac{\pi}{2}^-} = (\cos x)^{\cos x} = \lim_{x=\frac{\pi}{2}^-} e^{\ln f(x)} = e^0 = 1$$

45. The limit leads to the indeterminate form 1^∞.

Let $f(x) = (1+x)^{1/x}$.

$$\ln(1+x)^{1/x} = \frac{\ln(1+x)}{x}$$

$$\lim_{x\to0^+} \frac{\ln(1+x)}{x} = \lim_{x\to0^+} \frac{\frac{1}{1+x}}{1} = 1$$

$$\lim_{x\to0^+} (1+x)^{1/x} = \lim_{x\to0^+} e^{\ln f(x)} = e^1 = e$$

46. The limit leads to the indeterminate form 0^0.

Let $f(x) = (\sin x)^{\tan x}$

$$\ln(\sin x)^{\tan x} = \tan x\ln(\sin x) = \frac{\ln(\sin x)}{\cot x}$$

$$\lim_{x\to0^+} \frac{\ln(\sin x)}{\cot x} = \lim_{x\to0^+} \frac{\frac{\cos x}{\sin x}}{-\csc^2 x} = \lim_{x\to0^+} (-\sin x\cos x) = 0$$

$$\lim_{x\to0^+} (\sin x)^{\tan x} = \lim_{x\to0^+} e^{\ln f(x)} = e^0 = 1$$

47. The limit leads to the indeterminate form $1^{-\infty}$.

Let $f(x) = x^{1/(1-x)}$.

$$\ln x^{1/(1-x)} = \frac{\ln x}{1-x}$$

$$\lim_{x\to1^+} \frac{\ln x}{1-x} = \lim_{x\to1^+} \frac{\frac{1}{x}}{-1} = -1$$

$$\lim_{x\to1^+} x^{1/(1-x)} = \lim_{x\to1^+} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

48. $$\int_x^{2x} \frac{dt}{t} = \Big[\ln|t|\Big]_x^{2x} = \ln|2x| - \ln|x| = \ln\left|\frac{2x}{x}\right|$$

$$\lim_{x\to\infty} \int_x^{2x} \frac{dt}{t} = \lim_{x\to\infty} \ln\left|\frac{2x}{x}\right| = \lim_{x\to\infty} \ln 2 = \ln 2$$

49. $$\lim_{x\to1} \frac{x^3-1}{4x^3-x-3} = \lim_{x\to1} \frac{3x^2}{12x^2-1} = \frac{3}{11}$$

50. $$\lim_{x\to\infty} \frac{2x^2+3x}{x^3+x+1} = \lim_{x\to\infty} \frac{4x+3}{3x^2+1} = \lim_{x\to\infty} \frac{4}{6x} = 0$$

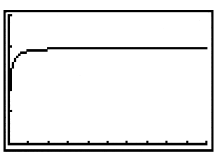
51. $$\lim_{x\to1} \frac{\int_1^x \cos t\,dt}{x^2-1} = \lim_{x\to1} \frac{\sin x - \sin 1}{x^2-1} = \lim_{x\to1} \frac{\cos x}{2x} = \frac{\cos 1}{2}$$

52. $$\lim_{x\to 1}\frac{\int_1^x \frac{dt}{t}}{x^3-1}=\lim_{x\to 1}\frac{\ln x-\ln 1}{x^3-1}=\lim_{x\to 1}\frac{\frac{1}{x}}{3x^2}=\frac{1}{3}$$

53. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{\left(\frac{9}{2}\right)(9x+1)^{-1/2}}{(1/2)(x+1)^{-1/2}}=\lim_{x\to\infty}\frac{9\sqrt{x+1}}{\sqrt{9x+1}}$$

(b)



[0, 100] by [0, 4]

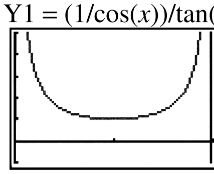
The limit appears to be 3.

(c) $$\lim_{x\to\infty}\frac{\sqrt{9x+1}}{\sqrt{x+1}}=\lim_{x\to\infty}\frac{\sqrt{9+\frac{1}{x}}}{\sqrt{1+\frac{1}{x}}}=\frac{\sqrt{9}}{\sqrt{1}}=3$$

54. **(a)** L'Hôpital's Rule does not help because applying L'Hôpital's Rule to this quotient essentially "inverts" the problem by interchanging the numerator and denominator (see below). It is still essentially the same problem and one is no closer to a solution. Applying L'Hôpital's Rule a second time returns to the original problem.

$$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\sec x\tan x}{\sec^2 x}=\lim_{x\to\pi/2}\frac{\tan x}{\sec x}$$

(b) Y1 = (1/cos(x))/tan(x)



[0, π] by [–1, 5]

The limit appears to be 1.

(c) $$\lim_{x\to\pi/2}\frac{\sec x}{\tan x}=\lim_{x\to\pi/2}\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}=\lim_{x\to\pi/2}\frac{1}{\sin x}=1$$

55. Find c such that $\lim_{x\to 0} f(x)=c$.

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{9x-3\sin 3x}{5x^3}=\lim_{x\to 0}\frac{9-9\cos 3x}{15x^2}=\lim_{x\to 0}\frac{27\sin 3x}{30x}=\lim_{x\to 0}\frac{81\cos 3x}{30}=\frac{81}{30}=\frac{27}{10}$$

Thus, $c=\frac{27}{10}$. This works since $\lim_{x\to 0} f(x)=c=f(0)$, so f is continuous.

56. $f(x)$ is defined at $x\neq 0$. $\lim_{x\to 0} f(x)$ leads to the indeterminate form 0^0.

$$\ln|x|^x=x\ln|x|=\frac{\ln|x|}{\frac{1}{x}}$$

$$\lim_{x\to 0}\frac{\ln|x|}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}}=\lim_{x\to 0}-x=0$$

$$\lim_{x\to 0}|x|^x=\lim_{x\to 0}e^{x\ln|x|}=e^0=1$$

Thus, f has a removable discontinuity at $x=0$. Extend the definition of f by letting $f(0)=1$.

57. **(a)** The limit leads to the indeterminate form 1^∞.

Let $f(k)=\left(1+\frac{r}{k}\right)^{kt}$.

$$\ln f(k)=kt\ln\left(1+\frac{r}{k}\right)=\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}$$

$$\begin{aligned}\lim_{k\to\infty}\frac{t\ln\left(1+\frac{r}{k}\right)}{\frac{1}{k}}&=\lim_{k\to\infty}\frac{t\left(-\frac{r}{k^2}\right)\left(1+\frac{r}{k}\right)^{-1}}{-\frac{1}{k^2}}\\&=\lim_{k\to\infty}\frac{rt}{1+\frac{r}{k}}\\&=\frac{rt}{1}\\&=rt\end{aligned}$$

$$\begin{aligned}\lim_{k\to\infty}A_0\left(1+\frac{r}{k}\right)^{kt}&=A_0\lim_{k\to\infty}\left(1+\frac{r}{k}\right)^{kt}\\&=A_0\lim_{k\to\infty}e^{\ln f(k)}\\&=A_0e^{rt}\end{aligned}$$

(b) Part (a) shows that as the number of compoundings per year increases toward infinity, the limit of interest compounded k times per year is interest compounded continuously.

58. **(a)** For $x\neq 0$, $\frac{f'(x)}{g'(x)}=\frac{1}{1}=1$.

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)}=1$$

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\frac{2}{1}=2$$

(b) This does not contradict L'Hôpital's Rule since $\lim_{x\to 0}f(x)=2$ and $\lim_{x\to 0}g(x)=1$.

59. **(a)**
$$\begin{aligned}A(t)&=\int_0^t e^{-x}dx\\&=[-e^{-x}]_0^t\\&=e^{-t}+1\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}A(t)&=\lim_{t\to\infty}(-e^{-t}+1)\\&=\lim_{t\to\infty}\left(-\frac{1}{e^t}+1\right)\\&=1\end{aligned}$$

(b)
$$\begin{aligned}V(t)&=\pi\int_0^t(e^{-x})^2\,dx\\&=\pi\int_0^t e^{-2x}\,dx\\&=\pi\left[-\frac{1}{2}e^{-2x}\right]_0^t\\&=\pi\left(-\frac{1}{2}e^{-2t}+\frac{1}{2}\right)\\&=\frac{\pi}{2}\left(-e^{-2t}+1\right)\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty}\frac{V(t)}{A(t)}&=\lim_{t\to\infty}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}\\&=\frac{\frac{\pi}{2}(1)}{1}\\&=\frac{\pi}{2}\end{aligned}$$

(c)
$$\begin{aligned}\lim_{t\to 0^+}\frac{V(t)}{A(t)}&=\lim_{t\to 0^+}\frac{\frac{\pi}{2}(-e^{-2t}+1)}{-e^{-t}+1}\\&=\lim_{t\to 0^+}\frac{\frac{\pi}{2}(2e^{-2t})}{e^{-t}}\\&=\frac{\frac{\pi}{2}(2)}{1}\\&=\pi\end{aligned}$$

60. **(a)**

x	$f(x)$
0.1	0.04542
0.01	0.00495
0.001	0.00050
0.0001	0.00005

The limit appears to be 0.

(b) $\lim_{x\to 0}\frac{\sin x}{1+2x}=\frac{0}{1}=0$

L'Hôpital's Rule is not applied here because the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, since the denominator has limit 1.

61. **(a)** $f(x)=e^{x\ln(1+1/x)}$

$1+\frac{1}{x}>0$ when $x<-1$ or $x>0$

Domain: $(-\infty,-1)\cup(0,\infty)$

(b) The form is 0^{-1}, so $\lim_{x\to -1^-} f(x) = \infty$.

(c) $$\lim_{x\to-\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to-\infty}\frac{\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$$
$$= \lim_{x\to-\infty}\frac{\left(-\frac{1}{x^2}\right)\left(1+\frac{1}{x}\right)^{-1}}{\frac{-1}{x^2}}$$
$$= \lim_{x\to-\infty}\frac{1}{1+\frac{1}{x}}$$
$$= 1$$
$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} e^{x\ln(1+1/x)} = e$$

62. False; need $g'(a) \neq 0$. Consider $f(x) = \sin^2 x$ and $g(x) = x^2$ with $a = 0$. Here $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$.

63. False; the limit is 1.

64. C; $\lim_{x\to 0}\frac{x}{\tan x} = \frac{1}{\sec^2 x} = \frac{1}{1} = 1$

65. D; $\lim_{x\to 1}\frac{1-\frac{1}{x}}{1-\frac{1}{x^2}} = \lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{2}{x^3}} = \lim_{x\to 1}\frac{x^3}{2x^2} = \frac{1}{2}$

66. B; $\lim_{x\to\infty}\frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty}\frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}} = \frac{\ln 3}{\ln 2}$

67. E; $\ln\left(1+\frac{1}{x}\right)^{3x} = 3x\ln\left(1+\frac{1}{x}\right) = \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{3x}}$

$$\lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{3x}}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cdot\frac{-1}{x^2}}{-\frac{1}{3x^2}}\right)$$
$$= \lim_{x\to\infty}\frac{3}{\left(1+\frac{1}{x}\right)}$$
$$= \frac{3}{1+0}$$
$$= 3$$
$$\lim_{x\to\infty} e^{\ln f(x)} = e^3$$

68. Possible answers:

(a) $f(x) = 7(x-3)$; $g(x) = x-3$
$$\lim_{x\to 3}\frac{f(x)}{g(x)} = \lim_{x\to 3}\frac{7(x-3)}{x-3} = \lim_{x\to 3}\frac{7}{1} = 7$$

(b) $f(x) = (x-3)^2$; $g(x) = x-3$
$$\lim_{x\to 3}\frac{f(x)}{g(x)} = \lim_{x\to 3}\frac{(x-3)^2}{x-3}$$
$$= \lim_{x\to 3}\frac{2(x-3)}{1}$$
$$= 0$$

(c) $f(x) = x-3$; $g(x) = (x-3)^3$
$$\lim_{x\to 3}\frac{f(x)}{g(x)} = \lim_{x\to 3}\frac{x-3}{(x-3)^3}$$
$$= \lim_{x\to 3}\frac{1}{3(x-3)^2}$$
$$= \infty$$

69. Answers may vary.

(a) $f(x) = 3x+1$; $g(x) = x$
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{3x+1}{x} = \lim_{x\to\infty}\frac{3}{1} = 3$$

(b) $f(x) = x+1$; $g(x) = x^2$
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x+1}{x^2} = \lim_{x\to\infty}\frac{1}{2x} = 0$$

(c) $f(x) = x^2$; $g(x) = x+1$
$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^2}{x+1} = \lim_{x\to\infty}\frac{2x}{1} = \infty$$

70. (a) Because the difference in the numerator is so small compared to the values being subtracted, any calculator or computer with limited precision will give the incorrect result that $1-\cos x^6$ is 0 for even moderately small values of x. For example, at $x = 0.1$, $\cos x^6 = 0.9999999999995$ (13 places), so on a 10-place calculator, $\cos x^6 = 1$ and $1-\cos x^6 = 0$.

(b) Same reason as in part (a) applies.

(c) $$\lim_{x\to 0}\frac{1-\cos x^6}{x^{12}}=\lim_{x\to 0}\frac{6x^5\sin x^6}{12x^{11}}$$
$$=\lim_{x\to 0}\frac{\sin x^6}{2x^6}$$
$$=\lim_{x\to 0}\frac{6x^5\cos x^6}{12x^5}$$
$$=\lim_{x\to 0}\frac{\cos x^6}{2}$$
$$=\frac{1}{2}$$

(d) The graph and/or table on a grapher show the value of the function to be 0 for x-values moderately close to 0, but the limit is $\frac{1}{2}$. The calculator is giving unreliable information because there is significant round-off error in computing values of this function on a limited precision device.

71. **(a)** $f'(x)=3x^2,\ g'(x)=2x-1$

$f(1)-f(-1)=2,\ g(1)-g(-1)=-2$

$$\frac{3c^2}{2c-1}=\frac{2}{-2}$$
$$3c^2=-2c+1$$
$$3c^2+2c-1=0$$
$$(3c-1)(c+1)=0$$
$$c=\frac{1}{3}\text{ or } c=-1$$

The value of c that satisfies the property is $c=\frac{1}{3}$.

(b) $f'(x)=-\sin x,\ g'(x)=\cos x$

$f\left(\frac{\pi}{2}\right)-f(0)=-1,\ g\left(\frac{\pi}{2}\right)-g(0)=1$

$$\frac{-\sin c}{\cos c}=\frac{-1}{1}$$
$$\tan c=1$$
$$c=\tan^{-1}1=\frac{\pi}{4}\text{ on }\left(0,\frac{\pi}{2}\right)$$

72. **(a)** $\ln f(x)^{g(x)}=g(x)\ln f(x)$

$$\lim_{x\to c}(g(x)\ln f(x))$$
$$=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$$
$$=\infty(-\infty)$$
$$=-\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{-\infty}=0$$

(b) $$\lim_{x\to c}\left(g(x)\ln f(x)\right)$$
$$=\left(\lim_{x\to c}g(x)\right)\left(\lim_{x\to c}\ln f(x)\right)$$
$$=(-\infty)(-\infty)$$
$$=\infty$$
$$\lim_{x\to c}f(x)^{g(x)}=\lim_{x\to c}e^{\ln f(x)^{g(x)}}=e^{\infty}=\infty$$

Quick Quiz Sections 9.1 and 9.2

1. C; $$\lim_{x\to 0}\frac{(x+1)^{4/3}-\left(\frac{4}{3}\right)x-1}{x^2}$$
$$=\lim_{x\to 0}\frac{\left(\frac{4}{3}\right)(x+1)^{1/3}-\left(\frac{4}{3}\right)}{2x}$$
$$=\lim_{x\to 0}\frac{\left(\frac{4}{9}\right)(x+1)^{-2/3}}{2}$$
$$=\frac{2}{9}$$

2. D; $\lim_{x\to 0^+}(3x^{2x})$

$$\ln x^{2x}=2x\ln x=\frac{2\ln x}{\frac{1}{x}}$$
$$\lim_{x\to 0^+}\frac{2\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{2/x}{-\frac{1}{x^2}}=\lim_{x\to 0^+}(-2x)=0$$
$$\lim_{x\to 0^+}3e^{\ln f(x)}=3e^0=3$$

3. B; $$\lim_{x\to 2}\frac{\int_2^x \sin t\,dt}{x^2-4}=\lim_{x\to 2}\frac{-\cos x+\cos 2}{x^2-4}$$
$$=\lim_{x\to 2}\frac{\sin x}{2x}$$
$$=\frac{\sin 2}{4}$$

4. **(a)** $\left(\dfrac{\frac{1}{2}}{-4}\right)^{1/3} = -\dfrac{1}{2}$

$\dfrac{-4}{-\frac{1}{2}} = 8$

(b) $-\dfrac{1}{2}$

(c) $a_n = 8\left(-\dfrac{1}{2}\right)^{n-1} = (-1)^{n-1}(2^{4-n})$

(d) $a_n = \left(-\dfrac{1}{2}\right)a_{n-1}$

Section 9.3 Relative Rates of Growth (pp. 457–462)

Exploration 1 Comparing Rates of Growth as $x \to \infty$

1. $\lim\limits_{x\to\infty} \dfrac{a^x}{x^2} = \lim\limits_{x\to\infty} \dfrac{(\ln a)(a^x)}{2x}$

$= \lim\limits_{x\to\infty} \dfrac{(\ln a)^2\ a^x}{2}$

$= \infty,$

so a^x grows faster than x^2 as $x \to \infty$.

2. $\lim\limits_{x\to\infty} \dfrac{3^x}{2^x} = \lim\limits_{x\to\infty} 1.5^x = \infty$

3. $\lim\limits_{x\to\infty} \dfrac{a^x}{b^x} = \lim\limits_{x\to\infty} \left(\dfrac{a}{b}\right)^x = \infty$ because $\dfrac{a}{b} > 1$.

Quick Review 9.3

1. $\lim\limits_{x\to\infty} \dfrac{\ln x}{e^x} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}}{e^x} = 0$

2. $\lim\limits_{x\to\infty} \dfrac{e^x}{x^3} = \lim\limits_{x\to\infty} \dfrac{e^x}{3x^2} = \lim\limits_{x\to\infty} \dfrac{e^x}{6x} = \lim\limits_{x\to\infty} \dfrac{e^x}{6} = \infty$

3. $\lim\limits_{x\to-\infty} \dfrac{x^2}{e^{2x}} = \infty$

4. $\lim\limits_{x\to\infty} \dfrac{x^2}{e^{2x}} = \lim\limits_{x\to\infty} \dfrac{2x}{2e^{2x}} = \lim\limits_{x\to\infty} \dfrac{2}{4e^{2x}} = 0$

5. $-3x^4$

6. $\dfrac{2x^3}{x} = 2x^2$

7. $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{x+\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1+\frac{1}{x}}{1} = 1$

8. $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+5x}}{2x}$

$= \lim\limits_{x\to\infty} \sqrt{1+\dfrac{5}{4x}}$

$= 1$

9. **(a)** $f(x) = \dfrac{e^x + x^2}{e^x} = 1 + \dfrac{x^2}{e^x}$

$f'(x) = \dfrac{2xe^x - x^2e^x}{e^{2x}} = \dfrac{2x - x^2}{e^x}$

$\dfrac{2x - x^2}{e^x} = 0$

$x(2-x) = 0$

$x = 0$ or $x = 2$

$f'(x) < 0$ for $x < 0$ or $x > 2$

The graph decreases, increases, and then decreases.

$f(0) = 1;\ f(2) = 1 + \dfrac{4}{e^2} \approx 1.541$

f has a local maximum at $\approx(2, 1.541)$ and has a local minimum at $(0, 1)$.

(b) f is increasing on $[0, 2]$.

(c) f is decreasing on $(-\infty, 0]$ and $[2, \infty)$.

10. $f(x) = \dfrac{x+\sin x}{x} = 1 + \dfrac{\sin x}{x},\ x \neq 0$

Observe that $\left|\dfrac{\sin x}{x}\right| < 1$ since $|\sin x| < |x|$ for $x \neq 0$.

$\lim\limits_{x\to 0} f(x) = 1 + \lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1 + 1 = 2$

Thus the values of f get close to 2 as x gets close to 0, so f doesn't have an absolute maximum value since f is not defined at 0.

Section 9.3 Exercises

1. $$\lim_{x\to\infty}\frac{e^x}{x^3-3x+1}=\lim_{x\to\infty}\frac{e^x}{3x^2-3}=\lim_{x\to\infty}\frac{e^x}{6x}=\lim_{x\to\infty}\frac{e^x}{6}=\infty$$

2. $$\lim_{x\to\infty}\frac{e^x}{x^{20}}=\lim_{x\to\infty}\frac{e^x}{20!}=\infty$$

3.
$$\lim_{x\to\infty}\frac{e^x}{e^{\cos x}}=\lim_{x\to\infty}\frac{e^x}{(-\sin x)e^{\cos x}},$$
$-1\le\cos x\le 1,$
$$\lim_{x\to\infty}\frac{e^x}{(-\sin x)e^{\cos x}}=\infty$$

4. $$\lim_{x\to\infty}\frac{e^x}{(5/2)^x}=\lim_{x\to\infty}\left(\frac{2e}{5}\right)^x=\infty \text{ since } \frac{2e}{5}>1.$$

5. $$\lim_{x\to\infty}\frac{\ln x}{x-\ln x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{1-\left(\frac{1}{x}\right)}=\lim_{x\to\infty}\frac{1}{x-1}=0$$

6. $$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\left(\frac{1}{2}\right)(x)^{-1/2}}=\lim_{x\to\infty}\frac{2}{(x)^{-1/2}x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

7. $$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\left(\frac{1}{3}\right)(x)^{-2/3}}=\lim_{x\to\infty}\frac{3}{(x)^{-2/3}x}=\lim_{x\to\infty}\frac{3}{\sqrt[3]{x}}=0$$

8. $$\lim_{x\to\infty}\frac{\ln x}{x^3}=\lim_{x\to\infty}\frac{\frac{1}{x}}{3x^2}=\lim_{x\to\infty}\frac{1}{3x^3}=0$$

9. $$\lim_{x\to\infty}\frac{x^2+4x}{x^2}=\lim_{x\to\infty}\frac{2x+4}{2x}=\lim_{x\to\infty}\frac{2}{2}=1$$

10. $$\lim_{x\to\infty}\frac{\sqrt{x^4+5x}}{x^2}=\lim_{x\to\infty}\sqrt{\frac{x^4+5x}{x^4}}=\lim_{x\to\infty}\sqrt{1+\frac{5}{x^3}}=1$$

11. $$\lim_{x\to\infty}\frac{(x^6+x^2)^{1/3}}{x^2}=\lim_{x\to\infty}\left(\frac{x^6+x^2}{x^6}\right)^{1/3}=\lim_{x\to\infty}\sqrt[3]{1+\frac{1}{x^4}}=1$$

12. $$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2}=\lim_{x\to\infty}\frac{2x+\cos x}{2x}=\lim_{x\to\infty}\frac{2x}{2x}+\lim_{x\to\infty}\frac{\cos x}{2x}=1+0=1,$$
since $-1\le\cos x\le 1$.

13. $$\lim_{x\to\infty}\frac{\log\sqrt{x}}{\ln x}=\lim_{x\to\infty}\frac{\frac{1}{2x\ln 10}}{\frac{1}{x}}=\frac{1}{2\ln 10}$$

14. $$\lim_{x\to\infty}\frac{e^{x+1}}{e^x}=e$$

15. First observe that $\sqrt{1+x^4}$ grows at the same rate as x^2.
$$\lim_{x\to\infty}\frac{\sqrt{1+x^4}}{x^2}=\lim_{x\to\infty}\sqrt{\frac{1+x^4}{x^4}}=\lim_{x\to\infty}\sqrt{\frac{1}{x^4}+1}=1$$
Next compare x^2 with e^x.
$$\lim_{x\to\infty}\frac{x^2}{e^x}=\lim_{x\to\infty}\frac{2x}{e^x}=\lim_{x\to\infty}\frac{2}{e^x}=0$$
x^2 grows more slowly than e^x as $x\to\infty$, so $\sqrt{1+x^4}$ grows more slowly than e^x as $x\to\infty$.

16. $\lim_{x\to\infty} \frac{4^x}{e^x} = \lim_{x\to\infty} \left(\frac{4}{e}\right)^x = \infty$ since $\frac{4}{e} > 1$.

4^x grows more quickly than e^x as $x \to \infty$.

17. $\lim_{x\to\infty} \frac{x\ln x - x}{e^x} = \lim_{x\to\infty} \frac{x\left(\frac{1}{x}\right)+\ln x - 1}{e^x}$

$= \lim_{x\to\infty} \frac{\ln x}{e^x}$

$= \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x}$

$= 0$

$x \ln x - x$ grows more slowly than e^x as $x \to \infty$.

18. $\lim_{x\to\infty} \frac{xe^x}{e^x} = \lim_{x\to\infty} x = \infty$

xe^x grows more quickly than e^x as $x \to \infty$.

19. $\lim_{x\to\infty} \frac{x^{1000}}{e^x} = 0$ (Repeated application of L'Hôpital's Rule gets $\lim_{x\to\infty} \frac{1000!}{e^x} = 0.$)

x^{1000} grows slower than e^x as $x \to \infty$.

20. $\lim_{x\to\infty} \frac{\frac{(e^x+e^{-x})}{2}}{e^x} = \lim_{x\to\infty} \left(\frac{1}{2} + \frac{1}{2e^{2x}}\right) = \frac{1}{2}$

$\frac{e^x + e^{-x}}{2}$ grows at the same rate as e^x as $x \to \infty$.

21. $\lim_{x\to\infty} \frac{x^3+3}{x^2} = \lim_{x\to\infty} \left(x + \frac{3}{x^2}\right) = \infty$

$x^3 + 3$ grows more quickly than x^2 as $x \to \infty$.

22. $\lim_{x\to\infty} \frac{15x+3}{x^2} = \lim_{x\to\infty} \left(\frac{15}{x} + \frac{3}{x^2}\right) = 0$

$15x + 3$ grows more slowly than x^2 as $x \to \infty$.

23. $\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$

$\ln x$ grows more slowly than x^2 as $x \to \infty$.

24. $\lim_{x\to\infty} \frac{2^x}{x^2} = \lim_{x\to\infty} \frac{(\ln 2)2^x}{2x} = \lim_{x\to\infty} \frac{(\ln 2)^2\, 2^x}{2} = \infty.$

2^x grows more quickly than x^2 as $x \to \infty$.

25. $\lim_{x\to\infty} \frac{\log_2 x^2}{\ln x} = \lim_{x\to\infty} \frac{2\log_2 x}{\ln x}$

$= \lim_{x\to\infty} \frac{2\frac{\ln x}{\ln 2}}{\ln x}$

$= \frac{2}{\ln 2}$

$\log_2 x^2$ grows at the same rate as $\ln x$ as $x \to \infty$.

26. $\lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}}{\ln x} = \lim_{x\to\infty} \frac{1}{\sqrt{x}\ln x} = 0$

$\frac{1}{\sqrt{x}}$ grows more slowly than $\ln x$ as $x \to \infty$.

27. $\lim_{x\to\infty} \frac{e^{-x}}{\ln x} = \lim_{x\to\infty} \frac{1}{e^x \ln x} = 0$

e^{-x} grows more slowly than $\ln x$ as $x \to \infty$.

28. $\lim_{x\to\infty} \frac{5\ln x}{\ln x} = 5$

$5\ln x$ grows at the same rate as $\ln x$ as $x \to \infty$.

29. Compare e^x to x^x.

$\lim_{x\to\infty} \frac{e^x}{x^x} = \lim_{x\to\infty} \left(\frac{e}{x}\right)^x = 0$

e^x grows more slowly than x^x.

Compare e^x to $(\ln x)^x$.

$\lim_{x\to\infty} \frac{e^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{e}{\ln x}\right)^x = 0$

e^x grows more slowly than $(\ln x)^x$.

Compare e^x to $e^{x/2}$.

$\lim_{x\to\infty} \frac{e^x}{e^{x/2}} = \lim_{x\to\infty} e^{x/2} = \infty$

e^x grows more quickly than $e^{x/2}$.

Compare x^x to $(\ln x)^x$.

$\lim_{x\to\infty} \frac{x^x}{(\ln x)^x} = \lim_{x\to\infty} \left(\frac{x}{\ln x}\right)^x = \infty$ since

$$\lim_{x\to\infty}\frac{x}{\ln x}=\lim_{x\to\infty}\frac{1}{\frac{1}{x}}=\infty.$$

x^x grows more quickly than $(\ln x)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $e^{x/2}$, e^x, $(\ln x)^x$, x^x.

30. Compare 2^x to x^2.

$$\lim_{x\to\infty}\frac{2^x}{x^2}=\lim_{x\to\infty}\frac{(\ln 2)2^x}{2x}=\lim_{x\to\infty}\frac{(\ln 2)^2\,2^x}{2}=\infty$$

2^x grows more quickly than x^2.

Compare 2^x to $(\ln 2)^x$.

$$\lim_{x\to\infty}\frac{2^x}{(\ln 2)^x}=\lim_{x\to\infty}\left(\frac{2}{\ln 2}\right)^x=\infty \text{ since}$$

$\frac{2}{\ln 2}>1.$

2^x grows more quickly than $(\ln 2)^x$.

Compare 2^x to e^x.

$$\lim_{x\to\infty}\frac{2^x}{e^x}=\lim_{x\to\infty}\left(\frac{2}{e}\right)^x=0 \text{ since } \frac{2}{e}<1.$$

2^x grows more slowly than e^x.

Compare x^2 to $(\ln 2)^x$.

$$\lim_{x\to\infty}\frac{x^2}{(\ln 2)^x}=\infty \text{ since } \lim_{x\to\infty}x^2=\infty \text{ and}$$

$\lim_{x\to\infty}(\ln 2)^x=0.$

x^2 grows more quickly than $(\ln 2)^x$.

Thus, in order from slowest-growing to fastest-growing, we get $(\ln 2)^x$, x^2, 2^x, e^x.

31. Compare f_1 to f_2.

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{10x+1}}{\sqrt{x}}=\lim_{x\to\infty}\sqrt{10+\frac{1}{x}}=\sqrt{10}$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$$\lim_{x\to\infty}\frac{f_3(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{x+1}}{\sqrt{x}}=\lim_{x\to\infty}\sqrt{1+\frac{1}{x}}=1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

32. Compare f_1 to f_2.

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{x^4+x}}{x^2}=\lim_{x\to\infty}\sqrt{1+\frac{1}{x^3}}=1$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 to f_3.

$$\lim_{x\to\infty}\frac{f_3(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{x^4-x^3}}{x^2}=\lim_{x\to\infty}\sqrt{1-\frac{1}{x}}=1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

33. Compare f_1 to f_2.

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{9^x+2^x}}{3^x}=\lim_{x\to\infty}\frac{\sqrt{9^x+2^x}}{\sqrt{9^x}}=\lim_{x\to\infty}\sqrt{1+\left(\frac{2}{9}\right)^x}=1$$

Thus f_1 to f_2 grow at the same rate.

Compare f_1 to f_3.

$$\lim_{x\to\infty}\frac{f_3(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\sqrt{9^x-4^x}}{3^x}=\lim_{x\to\infty}\frac{\sqrt{9^x-4^x}}{\sqrt{9^x}}=\lim_{x\to\infty}\sqrt{1-\left(\frac{4}{9}\right)^x}=1$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate as $x\to\infty$.

34. Compare f_1 to f_2.

$$\lim_{x\to\infty}\frac{f_2(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\frac{x^4+2x^2-1}{x+1}}{x^3}$$
$$=\lim_{x\to\infty}\frac{x^4+2x^2-1}{x^4+x^3}$$
$$=\lim_{x\to\infty}\frac{1+\frac{2}{x^2}-\frac{1}{x^4}}{1+\frac{1}{x}}$$
$$=1$$

Thus f_1 and f_2 grow at the same rate.

Compare f_1 and f_3.

$$\lim_{x\to\infty}\frac{f_3(x)}{f_1(x)}=\lim_{x\to\infty}\frac{\frac{2x^5-1}{x^2+1}}{x^3}$$
$$=\lim_{x\to\infty}\frac{2x^5-1}{x^5+x^3}$$
$$=\lim_{x\to\infty}\frac{2-\frac{1}{x^5}}{1+\frac{1}{x^2}}$$
$$=2$$

Thus f_1 and f_3 grow at the same rate.

By transitivity, f_2 and f_3 grow at the same rate, so all three functions grow at the same rate.

35. f grows more quickly than g.

36. g grows more quickly than f.

37. f and g grow at the same rate.

38. f and g grow at the same rate.

39. **(a)** The nth derivation of x^n is $n!$, a constant. We can apply L'Hôpital's Rule n times to find $\lim\limits_{x\to\infty}\frac{e^x}{x^n}$.

$$\lim_{x\to\infty}\frac{e^x}{x^n}=\cdots=\lim_{x\to\infty}\frac{e^x}{n!}=\infty$$

Thus e^x grows more quickly than x^n as $x\to\infty$ for any positive integer n.

(b) The nth derivative of a^x. $a>1$, is $(\ln a)^n a^x$. We can apply L' Hôpital's Rule n times to find $\lim\limits_{x\to\infty}\frac{a^x}{x^n}$.

$$\lim_{x\to\infty}\frac{a^x}{x^n}=\cdots=\lim_{x\to\infty}\frac{(\ln a)^n a^x}{n!}=\infty$$

Thus a^x grows more quickly than x^n as $x\to\infty$ for any positive integer n.

40. **(a)** Apply L'Hôpital's Rule n times to find

$$\lim_{x\to\infty}\frac{e^x}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}.$$

$$\lim_{x\to\infty}\frac{e^x}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}$$
$$=\cdots$$
$$=\lim_{x\to\infty}\frac{e^x}{a_nn!}$$
$$=\infty$$

Thus e^x grows more quickly than $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ as $x\to\infty$.

(b) Apply L'Hôpital's Rule n times to find

$$\lim_{x\to\infty}\frac{a^x}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}.$$

$$\lim_{x\to\infty}\frac{a^x}{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0}$$
$$=\cdots$$
$$=\lim_{x\to\infty}\frac{(\ln a)^n a^x}{a_nn!}$$
$$=\infty$$

Thus a^x grows more quickly than $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ as $x\to\infty$.

41. **(a)**
$$\lim_{x\to\infty}\frac{\ln x}{x^{1/n}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{n}x^{(1/n)-1}}$$
$$=\lim_{x\to\infty}\frac{n}{x^{1/n}}$$
$$=0$$

Thus $\ln x$ grows more slowly than $x^{1/n}$ as $x\to\infty$ for any positive integer n.

(b)
$$\lim_{x\to\infty}\frac{\ln x}{x^a}=\lim_{x\to\infty}\frac{\frac{1}{x}}{ax^{a-1}}=\lim_{x\to\infty}\frac{1}{ax^a}=0$$

Thus $\ln x$ grows more slowly than x^a as $x\to\infty$ for any number $a>0$.

42. $\lim_{x\to\infty} \dfrac{\ln x}{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}$

$= \lim_{x\to\infty} \dfrac{\frac{1}{x}}{na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1}$

$= \lim_{x\to\infty} \dfrac{1}{na_n x^{n} + (n-1)a_{n-1}x^{n-1} + \cdots + a_1 x}$

$= 0$

Thus $\ln x$ grows more slowly than any nonconstant polynomial as $x \to \infty$.

43. Compare $n\log_2 n$ to $n^{3/2}$ as $n \to \infty$.

$$\lim_{n\to\infty} \frac{n\log_2 n}{n^{3/2}} = \lim_{n\to\infty} \frac{\log_2 n}{n^{1/2}} = \lim_{n\to\infty} \frac{\frac{\ln n}{\ln 2}}{n^{1/2}} = \lim_{n\to\infty} \frac{\frac{1}{n\ln 2}}{\frac{1}{2n^{1/2}}} = \lim_{n\to\infty} \frac{2}{n^{1/2}(\ln 2)} = 0$$

Thus $n\log_2 n$ grows more slowly than $n^{3/2}$ as $n \to \infty$.

Compare $n\log_2 n$ to $n(\log_2 n)^2$.

$$\lim_{n\to\infty} \frac{n\log_2 n}{n(\log_2 n)^2} = \lim_{n\to\infty} \frac{1}{\log_2 n} = 0$$

Thus $n\log_2 n$ grows more slowly than $n(\log_2 n)^2$ as $n \to \infty$.

The algorithm of order of $n\log_2 n$ is likely the most efficient because of the three functions, it grows the most slowly as $n \to \infty$.

44. **(a)** It might take 1,000,000 searches if it is the last item in the search.

(b) $\log_2 1{,}000{,}000 \approx 19.9$; it might take 20 binary searches.

45. **(a)** The limit will be the ratio of the leading coefficients of the polynomials since the polynomials must have the same degree.

(b) By the same reason as in (a), the limit will be the ratio of the leading coefficients of the polynomial.

46. True; because $\lim_{n\to\infty} \dfrac{n\log_2 n}{n^{3/2}} = 0$.

47. False; they grow at the same rate.

48. E; $\lim_{x\to\infty} \dfrac{x^6+1}{x^5+x^2+1} = \lim_{x\to\infty} \dfrac{6!\,x}{5!} = \infty$

49. E; $\lim_{x\to\infty} \dfrac{x\ln x}{\log_{13} x} = \lim_{x\to\infty} \dfrac{x\ln x}{\frac{\ln x}{\ln 13}} = \lim_{x\to\infty} x\ln 13 = \infty$

50. C; $\lim_{x\to\infty} \dfrac{e^{x+2}}{e^x} = e^2$

51. D; $\lim_{x\to\infty} \dfrac{\sqrt{x^8+x^4}}{x^4} = \lim_{x\to\infty} \sqrt{\dfrac{x^8+x^4}{x^8}} = \lim_{x\to\infty} \sqrt{1+\dfrac{1}{x^4}} = 1$

52. **(a)** $\lim_{x\to\infty} \dfrac{x^5}{x^2} = \lim_{x\to\infty} x^3 = \infty$

x^5 grows more quickly than x^2.

(b) $\lim_{x\to\infty} \dfrac{5x^3}{2x^3} = \lim_{x\to\infty} \dfrac{5}{2} = \dfrac{5}{2}$

$5x^3$ and $2x^3$ have the same rate of growth.

(c) $m > n$ since $\lim_{x\to\infty} \dfrac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n} = \infty$.

(d) $m = n$ since $\lim_{x\to\infty} \dfrac{x^m}{x^n} = \lim_{x\to\infty} x^{m-n}$ is nonzero and finite.

(e) Degree of $g >$ degree of $f(m > n)$ since $\lim_{x\to\infty} \dfrac{g(x)}{f(x)} = \infty$.

(f) Degree of $g =$ degree of $f(m = n)$ since $\lim_{x\to\infty} \dfrac{g(x)}{f(x)}$ is nonzero and finite.

53. **(a)** $\lim_{x\to\infty} \dfrac{|f(x)|}{|g(x)|} = \lim_{x\to\infty} \dfrac{-f(x)}{-g(x)} = \lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$

Thus $|f|$ grows more quickly than $|g|$ as $x \to \infty$ by definition.

(b) $\lim_{x\to\infty}\frac{|f(x)|}{|g(x)|}=\lim_{x\to\infty}\frac{-f(x)}{-g(x)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}=L$

Thus $|f|$ grows at the same rate as $|g|$ as $x\to\infty$ by definition.

54. (a) $\lim_{x\to\infty}\frac{f(-x)}{g(-x)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$

Thus $f(-x)$ grows faster than $g(-x)$ by definition.

(b) $\lim_{x\to\infty}\frac{f(-x)}{g(-x)}=\lim_{x\to\infty}\frac{f(x)}{g(x)}=L$

Thus $f(-x)$ grows at the same rate as $g(-x)$ by definition.

Section 9.4 Improper Integrals (pp. 463–473)

Exploration 1 Investigating $\int_0^1\frac{dx}{x^p}$

1. Because $\frac{1}{x^p}$ has an infinite discontinuity at $x=0$.

2. $$\begin{aligned}\int_0^1\frac{dx}{x}&=\lim_{c\to0^+}\int_c^1\frac{dx}{x}\\&=\lim_{c\to0^+}[\ln x]_c^1\\&=\lim_{c\to0^+}(-\ln c)\\&=\infty\end{aligned}$$

3. If $p>1$, then $$\begin{aligned}\int_0^1\frac{dx}{x^p}&=\lim_{c\to0^+}\int_c^1\frac{dx}{x^p}\\&=\lim_{c\to0^+}\left.\frac{x^{-p+1}}{-p+1}\right|_c^1\\&=\lim_{c\to0^+}\left(\frac{1-c^{-p+1}}{-p+1}\right)\\&=\infty\end{aligned}$$
because $(-p+1)<0$.

4. If $0<p<1$, then
$$\begin{aligned}\int_x^1\frac{dx}{x^p}&=\lim_{c\to0^+}\int_c^1\frac{dx}{x^p}\\&=\lim_{c\to0^+}\left.\frac{x^{-p+1}}{-p+1}\right|_c^1\\&=\lim_{c\to0^+}\left(\frac{1-c^{-p+1}}{-p+1}\right)\\&=\frac{1}{1-p}\end{aligned}$$
because $-p+1>0$.

Exploration 2 Gabriel's Horn

1. The integral in Example 2, $\int_1^\infty\frac{dx}{x}$, represents the area of R. Since the integral diverges, region R has infinite area.

2. The volume is $$\begin{aligned}V&=\int_1^\infty2\pi\left(\frac{1}{x}\right)^2dx\\&=\lim_{c\to\infty}\int_1^c2\pi x^{-2}dx\\&=\lim_{c\to\infty}2\pi[-x^{-1}]_1^c\\&=\lim_{c\to\infty}2\pi\left[-\frac{1}{c}+1\right]\\&=2\pi.\end{aligned}$$

3. The area of the shadow would be twice the area of region R. Since R has infinite area, so does the shadow.

4. The integral for the volume converges, so the volume is finite. An integral representing the area of the shadow would diverge, so its area is infinite.

Quick Review 9.4

1. $\int_0^3\frac{dx}{x+3}=\Big[\ln|x+3|\Big]_0^3=\ln 6-\ln 3=\ln 2$

2. $$\begin{aligned}\int_{-1}^1\frac{x\,dx}{x^2+1}&=\left[\frac{1}{2}\ln\left|x^2+1\right|\right]_{-1}^1\\&=\frac{1}{2}\ln 2-\frac{1}{2}\ln 2\\&=0\end{aligned}$$

3. $\displaystyle\int \frac{dx}{x^2+4} = \frac{1}{4}\int \frac{dx}{\left(\frac{x}{2}\right)^2+1}$

$\displaystyle = \frac{1}{4}\left(2\tan^{-1}\frac{x}{2}\right)+C$

$\displaystyle = \frac{1}{2}\tan^{-1}\frac{x}{2}+C$

4. $\displaystyle\int \frac{dx}{x^4} = \int x^{-4}\,dx = -\frac{1}{3}x^{-3}+C$

5. $9-x^2>0$ for $-3<x<3$
The domain is $(-3, 3)$.

6. $x-1>0$ for $x>1$
The domain is $(1, \infty)$.

7. $-1 \le \cos x \le 1$, so $|\cos x| \le 1$.

$\displaystyle\left|\frac{\cos x}{x^2}\right| = \frac{|\cos x|}{|x|^2} \le \frac{1}{x^2}$

8. $x^2-1 \le x^2$ so $\sqrt{x^2-1} \le \sqrt{x^2} = x$ for $x>1$

$\displaystyle\frac{1}{\sqrt{x^2-1}} \ge \frac{1}{x}$

9. $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{4e^x-5}{3e^x+7}$

$\displaystyle = \lim_{x\to\infty}\frac{4e^x}{3e^x}$

$\displaystyle = \lim_{x\to\infty}\frac{4}{3}$

$\displaystyle = \frac{4}{3}$

Thus f and g grow at the same rate as $x \to \infty$.

10. $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{\sqrt{2x-1}}{\sqrt{x+3}}$

$\displaystyle = \lim_{x\to\infty}\sqrt{\frac{2x-1}{x+3}}$

$\displaystyle = \lim_{x\to\infty}\sqrt{\frac{2-\frac{1}{x}}{1+\frac{3}{x}}}$

$= \sqrt{2}$

Section 9.4 Exercises

1. **(a)** $\displaystyle\int_0^{\infty}\frac{2x}{x^2+1}\,dx = \lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}\,dx$

(b) $\lim_{b\to\infty}\int_0^b \frac{2x}{x^2+1}dx = \lim_{b\to\infty}(\ln(x^2+1))\Big|_0^b = \infty$

diverges

2. (a) $\int_1^\infty \frac{dx}{x^{1/3}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{x^{1/3}} = \lim_{b\to\infty}\left(\frac{3}{2}x^{2/3}\right)\Big|_1^b = \infty$

diverges

3. (a) $\int_{-\infty}^\infty \frac{2x}{(x^2+1)^2}dx$

$= \lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx$

(b) $\lim_{b\to-\infty}\int_b^0 \frac{2x}{(x^2+1)^2}dx + \lim_{b\to\infty}\int_0^b \frac{2x}{(x^2+1)^2}dx = \lim_{b\to-\infty}\left(\frac{-1}{x^2+1}\right)\Big|_b^0 + \lim_{b\to\infty}\left(\frac{-1}{x^2+1}\right)\Big|_0^b$

$= 0$

converges

4. (a) $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}}$

(b) $\lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\left(2\sqrt{x}\right)\Big|_1^b = \infty$

diverges

5. $\int_1^\infty \frac{dx}{x^4} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^4} = \lim_{b\to\infty}\left(\frac{-1}{3x^3}\right)\Big|_1^b = \frac{1}{3}$

6. $\int_1^\infty \frac{2dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{2dx}{x^3} = \lim_{b\to\infty}\left(\frac{-2}{2x^2}\right)\Big|_1^b = 1$

7. $\int_1^\infty \frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[3]{x}} = \lim_{b\to\infty}\left(\frac{3}{2}(x)^{2/3}\right)\Big|_1^b = \infty$

diverges

8. $\int_1^\infty \frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\int_1^b \frac{dx}{\sqrt[4]{x}} = \lim_{b\to\infty}\left(\frac{4}{3}(x)^{3/4}\right)\Big|_1^b = \infty$

diverges

9. $\int_{-\infty}^{-1} \frac{dx}{x^2} = \lim_{b\to-\infty}\int_b^{-1} \frac{dx}{x^2} = \lim_{b\to-\infty}\left(-\frac{1}{x}\right)\Big|_b^{-1} = 1$

10. $\int_{-\infty}^{0} \frac{dx}{(x-2)^3} = \lim_{b\to-\infty} \int_b^0 \frac{dx}{(x-2)^3}$
$= \lim_{b\to-\infty} \left(\frac{-1}{2(x-2)^2} \right)\Big|_b^0$
$= -\frac{1}{8}$

11. $\int_{-\infty}^{-2} \frac{2dx}{x^2-1} = \lim_{b\to-\infty} \int_b^{-2} \left[\frac{2}{x^2-1} \right] dx$
$= \lim_{b\to-\infty} \int_b^{-2} \left[\frac{1}{x-1} - \frac{1}{x+1} \right] dx$
$= \lim_{b\to-\infty} \left(\ln\left|\frac{x-1}{x+1}\right| \right)\Big|_b^{-2}$
$= \lim_{b\to-\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right| \right)$
$= \ln 3$

12. $\int_2^{\infty} \frac{3dx}{x^2-x} = \lim_{b\to\infty} \int_2^b \left[\frac{3}{x^2-x} \right] dx$
$= \lim_{b\to\infty} \int_2^b \left[\frac{3}{x-1} - \frac{3}{x} \right] dx$
$= \lim_{b\to\infty} \left(3\ln\left|\frac{x-1}{x}\right| \right)\Big|_2^b$
$= \lim_{b\to\infty} \left(3\ln\left|\frac{b-1}{b}\right| - 3\ln\left(\frac{1}{2}\right) \right)$
$= 3\ln 2$

13. $\int_{-1}^{\infty} \frac{dx}{x^2+5x+6}$
$= \lim_{b\to\infty} \int_{-1}^b \left[\frac{1}{x^2+5x+6} \right] dx$
$= \lim_{b\to\infty} \int_{-1}^b \left[\frac{1}{x+2} - \frac{1}{x+3} \right] dx$
$= \lim_{b\to\infty} \left(\ln\left|\frac{x+2}{x+3}\right| \right)\Big|_{-1}^b$
$= \lim_{b\to\infty} \left(\ln\left|\frac{b+2}{b+3}\right| - \ln\left(\frac{1}{2}\right) \right)$
$= \ln 2$

14. $\int_{-\infty}^{0} \frac{2dx}{x^2-4x+3}$
$= \lim_{b\to-\infty} \int_b^0 \left[\frac{2}{x^2-4x+3} \right] dx$
$= \lim_{b\to-\infty} \int_b^0 \left[\frac{1}{x-3} - \frac{1}{x-1} \right] dx$
$= \lim_{b\to-\infty} \left(\ln\left|\frac{x-3}{x-1}\right| \right)\Big|_b^0$
$= \lim_{b\to-\infty} \left(\ln 3 - \ln\left|\frac{b-3}{b-1}\right| \right)$
$= \ln 3$

15. $\int_1^{\infty} \frac{5x+6}{x^2+2x} dx = \lim_{b\to\infty} \int_1^b \left[\frac{5x+6}{x^2+2x} \right] dx$
$= \lim_{b\to\infty} \int_1^b \left[\frac{3}{x} + \frac{2}{x+2} \right] dx$
$= \lim_{b\to\infty} \left(\ln\left|x^3(x+2)^2\right| \right)\Big|_1^b$
$= \infty$
diverges

16. $\int_{-2}^{-\infty} \frac{2\,dx}{x^2-2x} = \lim_{b\to-\infty} \int_{-2}^b \left[\frac{2}{x^2-2x} \right] dx$
$= \lim_{b\to-\infty} \int_{-2}^b \left[\frac{1}{x-2} - \frac{1}{x} \right] dx$
$= \lim_{b\to-\infty} \left(\ln\left|\frac{x-2}{x}\right| \right)\Big|_{-2}^b$
$= \lim_{b\to-\infty} \left(\ln\left|\frac{b-2}{b}\right| - \ln 2 \right)$
$= -\ln 2$

17. $\int_1^{\infty} x e^{-2x}\, dx$
$= \lim_{b\to\infty} \int_1^b xe^{-2x} dx$
$= \lim_{b\to\infty} \left(-\frac{x}{2} - \frac{1}{4} \right) e^{-2x} \Big|_1^b$
$= \lim_{b\to\infty} \left[\left(-\frac{b}{2} - \frac{1}{4} \right) e^{-2b} - \left(-\frac{3}{4} e^{-2} \right) \right]$
$= \frac{3}{4} e^{-2}$

18. $\int_{-\infty}^{0} x^2 e^x\,dx = \lim_{b\to-\infty}\int_b^0 x^2 e^x dx$

$$= \lim_{b\to-\infty}\left((x^2-2x+2)e^{-2x}\right)\Big|_b^0$$
$$= \lim_{b\to-\infty}\left(\frac{b^2-2b+2}{e^{2b}}-2\right)$$
$$= 2$$

19. $\int_1^\infty x\ln x\,dx = \lim_{b\to\infty}\int_1^b x\ln x\,dx$

$$= \lim_{b\to\infty}\left(\frac{x^2}{2}\ln 2-\frac{x^2}{4}\right)\Bigg|_1^b$$
$$= \infty$$

diverges

20. $\int_0^\infty (x+1)e^{-x}\,dx = \lim_{b\to\infty}\int_0^b (x+1)e^{-x}\,dx$

$$= \lim_{b\to\infty}\left((-x-2)e^{-x}\right)\Big|_0^b$$
$$= \lim_{b\to\infty}\left(\frac{-b-2}{e^b}-(-2)\right)$$
$$= 2$$

21. $\int_{-\infty}^\infty e^{-|x|}\,dx = \lim_{b\to-\infty}\int_b^0 e^x dx + \lim_{b\to\infty}\int_0^b e^{-x}dx$

$$= \lim_{b\to-\infty}(e^x)\Big|_b^0 + \lim_{b\to\infty}(-e^{-x})\Big|_0^b$$
$$= 1+[-(-1)]$$
$$= 2$$

22. $\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

$$= \lim_{b\to-\infty}\int_b^0 2xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b 2xe^{-x^2}\,dx$$
$$= \lim_{b\to-\infty}\left(-e^{-x^2}\right)\Big|_b^0 + \lim_{b\to\infty}\left(-e^{-x^2}\right)\Big|_0^b$$
$$= 1+(-1)$$
$$= 0.$$

23. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

$$= \lim_{b\to-\infty}\int_b^0 \frac{dx}{e^x+e^{-x}} + \lim_{b\to\infty}\int_0^b \frac{dx}{e^x+e^{-x}}$$
$$\lim_{b\to-\infty}\left(\tan^{-1}(e^x)\right)\Big|_b^0 + \lim_{b\to\infty}\left(\tan^{-1}(e^x)\right)\Big|_0^b$$
$$= [\tan^{-1}1-\tan^{-1}0]+\left[\lim_{b\to\infty}\tan^{-1}e^b-\tan^{-1}1\right]$$
$$= \left(\frac{\pi}{4}-0\right)+\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$
$$= \frac{\pi}{2}$$

24. $\int_{-\infty}^\infty e^{2x}\,dx = \lim_{b\to-\infty}\int_b^0 e^{2x}\,dx + \lim_{b\to\infty}\int_0^b e^{2x}\,dx$

$$= \lim_{b\to-\infty}\left(\frac{e^{2x}}{2}\right)\Bigg|_b^0 + \lim_{b\to\infty}\left(\frac{e^{2x}}{2}\right)\Bigg|_0^b$$
$$= \infty$$

diverges

25. **(a)** The integral has an infinite discontinuity at the interior point $x = 1$.

(b) $\int_0^2 \frac{dx}{1-x^2}$

$$= \lim_{b\to1^-}\int_0^b \frac{dx}{1-x^2} + \lim_{b\to1^+}\int_b^2 \frac{dx}{1-x^2}$$
$$= \lim_{b\to1^-}\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\Bigg|_0^b + \lim_{b\to1^+}\left(\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right)\Bigg|_b^2$$
$$= \infty$$

diverges

26. **(a)** The integral has an infinite discontinuity at the endpoint $x = 1$.

(b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}}$

$$= \lim_{b\to1^-}(\sin^{-1}(x))\Big|_0^b$$
$$= \frac{\pi}{2}$$

27. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx = \lim_{b\to 0^+}\int_b^1 \frac{x+1}{\sqrt{x^2+2x}}\,dx$

$= \lim_{b\to 0^+}\left(\sqrt{x^2+2x}\right)\Big|_b^1$

$= \sqrt{3}$

28. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx = \lim_{b\to 0^+}\int_b^4 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

$= \lim_{b\to 0^+}\left(-2e^{-\sqrt{x}}\right)\Big|_b^4$

$= 2 - 2e^{-2}$

29. **(a)** The integral has an infinite discontinuity at the endpoint $x = 0$.

(b) $\int_0^1 x\ln(x)\,dx = \lim_{b\to 0^+}\int_b^1 x\ln(x)\,dx$

$= \lim_{b\to 0^+}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right)\Big|_b^1$

$= -\frac{1}{4}$

30. **(a)** The integral has an infinite discontinuity at the interior point $x = 0$.

(b) $\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

$= \lim_{b\to 0^-}\int_{-1}^b \frac{dx}{\sqrt{-x}} + \lim_{b\to 0^+}\int_b^4 \frac{dx}{\sqrt{x}}$

$= \lim_{b\to 0^-}\left(-2\sqrt{-x}\right)\Big|_{-1}^b + \lim_{b\to 0^+}\left(2\sqrt{x}\right)\Big|_b^4$

$= (0-(-2)) + (4-0) = 6$

31. $0 \le \frac{1}{1+e^x} \le \frac{1}{e^x}$ on $[1, \infty)$, converges because $\int_1^\infty \frac{1}{e^x}\,dx$ converges.

32. $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ on $[1, \infty)$, converges because $\int_1^\infty \frac{1}{x^3}\,dx$ converges.

33. $0 \le \frac{1}{x} \le \frac{2+\cos x}{x}$ on $[\pi, \infty)$, diverges because $\int_\pi^\infty \frac{1}{x}\,dx$ diverges.

34. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$

$= 2\int_0^1 \frac{dx}{\sqrt{x^4+1}} + 2\int_1^\infty \frac{dx}{\sqrt{x^4+1}}$

First integral is a proper integral. Second integral converges because $0 \le \frac{1}{\sqrt{x^4+1}} \le \frac{1}{x^2}$ on $[1, \infty)$ and $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

35. $\int_0^{\ln 2} y^{-2}e^{1/y}\,dy = \lim_{b\to 0^+}\int_b^{\ln 2} y^{-2}e^{1/y}\,dy$

$= \lim_{b\to 0^+}(-e^{1/y})\Big|_b^{\ln 2}$

$= \infty$

diverges

36. $\int_0^4 \frac{dr}{\sqrt{4-r}} = \lim_{b\to 4^-}\int_0^b \frac{dr}{\sqrt{4-r}}$

$= \lim_{b\to 4^-}\left(-2\sqrt{4-r}\right)\Big|_0^b$

$= 4$

37. $$\int_0^\infty \frac{ds}{(1+s)\sqrt{s}} = \lim_{b\to 0^+}\int_b^1 \frac{ds}{(1+s)\sqrt{s}} + \lim_{b\to\infty}\int_1^b \frac{ds}{(1+s)\sqrt{s}}$$
$$= \lim_{b\to 0^+}\left(2\tan^{-1}\sqrt{s}\right)\Big|_b^1 + \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{s}\right)\Big|_1^b$$
$$= 2\tan^{-1}1 - \lim_{b\to 0^+}\left(2\tan^{-1}\sqrt{b}\right) + \lim_{b\to\infty}\left(2\tan^{-1}\sqrt{b}\right) - 2\tan^{-1}1$$
$$= \frac{\pi}{2} - 0 + \pi - \frac{\pi}{2}$$
$$= \pi$$

38. $$\int_1^2 \frac{du}{u\sqrt{u^2-1}} = \lim_{b\to 1^+}\int_b^2 \frac{du}{u\sqrt{u^2-1}}$$
$$= \lim_{b\to 1^+}\left(\tan^{-1}\sqrt{u^2-1}\right)\Big|_b^2$$
$$= \frac{\pi}{3}$$

39. $$\int_0^\infty \frac{16\tan^{-1}v}{1+v^2}\,dv = \lim_{b\to\infty}\int_0^b \frac{16\tan^{-1}v}{1+v^2}\,dv$$
$$= \lim_{b\to\infty}(8(\tan^{-1}v)^2)\Big|_0^b$$
$$= 2\pi^2$$

40. $$\int_{-\infty}^0 \theta e^\theta\,d\theta = \lim_{b\to-\infty}\int_b^0 \theta e^\theta\,d\theta$$
$$= \lim_{b\to-\infty}((\theta-1)e^\theta)\Big|_0^b$$
$$= -1$$

41. $$\int_0^2 \frac{dt}{1-t} = \lim_{b\to 1^-}\int_0^b \frac{dt}{1-t} + \lim_{b\to 1^+}\int_b^2 \frac{dt}{1-t}$$
$$= \lim_{b\to 1^-}\left(-\ln|1-t|\right)\Big|_0^b + \lim_{b\to 1^+}\left(-\ln|1-t|\right)\Big|_b^2$$
$$= \infty + \infty$$
diverges

42. $$\int_{-1}^1 \ln(|w|)\,dw = 2\lim_{b\to 0^+}\int_b^1 \ln(w)\,dw$$
$$= 2\lim_{b\to 0^+}[w\ln w - w]_b^1$$
$$= 2\lim_{b\to 0^+}[-1 + b(1-\ln b)]$$
$$= -2$$

43. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}\,dx$$

Integrate $\int \frac{\ln x}{x^2}\,dx$ by parts.

$u = \ln x \qquad dv = \frac{dx}{x^2}$

$du = \frac{1}{x}dx \qquad v = -\frac{1}{x}$

$$\int \frac{\ln x}{x^2} = -\frac{\ln x}{x} + \int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$\begin{aligned}\text{Area} &= \lim_{b\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b \\ &= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right] \\ &= 1\end{aligned}$$

$$\left(\text{Note that } \lim_{b\to\infty}\frac{\ln b}{b} = \lim_{b\to\infty}\frac{\frac{1}{b}}{1} = 0.\right)$$

44. For $x \ge 0$, $y \ge 0$ on $[1, \infty)$.

$$\text{Area} = \int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$$

Integrate $\int \frac{\ln x}{x}\,dx$ by letting $u = \ln x$, so $du = \frac{dx}{x}$.

$$\int \frac{\ln x}{x}\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

$$\text{Area} = \lim_{b\to\infty}\left[\frac{1}{2}(\ln x)^2\right]_1^b = \lim_{b\to\infty}\frac{1}{2}(\ln b)^2 = \infty$$

45. **(a)** Since f is even, $f(-x) = f(x)$. Let $u = -x$, $du = -dx$.

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx \\ &= \int_{\infty}^{0} f(-u)(-1)\,du + \int_0^{\infty} f(x)\,dx \\ &= \int_0^{\infty} f(u)du + \int_0^{\infty} f(x)\,dx \\ &= 2\int_0^{\infty} f(x)\,dx\end{aligned}$$

(b) Since f is odd, $f(-x) = -f(x)$. Let $u = -x$, then $du = -dx$.

$$\begin{aligned}&\int_{-\infty}^{\infty} f(x)\,dx \\ &= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx \\ &= \int_{\infty}^{0} f(-u)(-1)\,du + \int_0^{\infty} f(x)\,dx \\ &= \int_{\infty}^{0} -f(u)(-1)\,du + \int_0^{\infty} f(x)\,dx \\ &= \int_{\infty}^{0} f(u)\,du + \int_0^{\infty} f(x)\,dx \\ &= -\int_0^{\infty} f(u)\,du + \int_0^{\infty} f(x)\,dx \\ &= -L + L \\ &= 0\end{aligned}$$

46. **(a)**

$$\begin{aligned}\int_0^{\infty}\frac{2x\,dx}{x^2+1} &= \lim_{b\to\infty}\int_0^b \frac{2x\,dx}{x^2+1} \\ &= \lim_{b\to\infty}[\ln(x^2+1)]_0^b \\ &= \lim_{b\to\infty}\ln(b^2+1) \\ &= \infty\end{aligned}$$

Thus the integral diverges.

(b) Both $\int_0^{\infty}\frac{2x\,dx}{x^2+1}$ and $\int_{-\infty}^{0}\frac{2x\,dx}{x^2+1}$ must converge in order for $\int_{-\infty}^{\infty}\frac{2x\,dx}{x^2+1}$ to converge.

(c)

$$\begin{aligned}&\lim_{b\to\infty}\int_{-b}^{b}\frac{2x\,dx}{x^2+1} \\ &= \lim_{b\to\infty}[\ln(x^2+1)]_{-b}^{b} \\ &= \lim_{b\to\infty}[\ln(b^2+1) - \ln(b^2+1)] \\ &= \lim_{b\to\infty} 0 \\ &= 0.\end{aligned}$$

Note that $\frac{2x}{x^2+1}$ is an odd function so

$$\int_{-b}^{b}\frac{2x\,dx}{x^2+1} = 0.$$

(d) Although the limit in part (c) is correct, this is not a valid way to evaluate the indefinite integral. In order for the integral to converge, there must be finite areas in both directions (toward ∞ and toward $-\infty$). In this case, there are infinite areas in both directions.

47. By symmetry, find the perimeter of one side, say for $0 \le x \le 1$, $y \ge 0$.

$$y^{2/3} = 1 - x^{2/3}$$
$$y = (1 - x^{2/3})^{3/2}$$
$$\frac{dy}{dx} = \frac{3}{2}(1 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -x^{-1/3}(1 - x^{2/3})^{1/2}$$
$$\left(\frac{dy}{dx}\right)^2 = x^{-2/3}(1 - x^{2/3}) = (x^{-2/3} - 1)$$
$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{x^{-2/3}} = x^{-1/3}$$
$$\int_0^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\int_b^1 x^{-1/3}\,dx = \lim_{b\to 0^+}\left[\frac{3}{2}x^{2/3}\right]_b^1 = \lim_{b\to 0^+}\left[\frac{3}{2} - \frac{3}{2}b^{2/3}\right] = \frac{3}{2}$$

Thus, the perimeter is $4\left(\frac{3}{2}\right) = 6$.

48. False; see Theorem 6.

49. True; see Theorem 6.

50. C; $\displaystyle\int_1^\infty \frac{dx}{x^{1.01}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1.01}} = \lim_{b\to\infty}\left(-\frac{100}{x^{0.01}}\right)\Bigg|_1^b = 100$

51. B; $\displaystyle\int_0^1 \frac{dx}{x^{0.5}} = \lim_{b\to 0^+}\int_b^1 \frac{dx}{x^{0.5}} = \lim_{b\to 0^+}\left(2\sqrt{x}\right)\Big|_b^1 = 2$

52. E; $\displaystyle\int_0^1 \frac{dx}{x-1} = \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-}\left(\ln(|x-1|)\right)\Big|_0^b = -\infty$

53. C; $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \lim_{b\to\infty}\int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty}(\tan^{-1} x)\Big|_0^b = \frac{\pi}{2}$

54. **(a)** $\displaystyle\int_1^\infty \frac{dx}{x^{0.5}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{0.5}} = \lim_{b\to\infty}\left(2\sqrt{x}\right)\Big|_1^b = \infty,$

or it diverges.

(b) $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty}\left(\ln|x|\right)\Big|_1^b = \infty,$

or it diverges.

(c) $\displaystyle\int_1^\infty \frac{dx}{x^{1.5}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^{1.5}} = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{x}}\right)\Bigg|_1^b = 2$

(d) $$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p} = \lim_{b\to\infty}\left(\frac{1}{1-p}\frac{1}{x^{1-p}}\right)\Bigg|_1^b = \lim_{b\to\infty}\left(\frac{1}{1-p}\frac{1}{b^{1-p}} - \frac{1}{1-p}\frac{1}{1^{1-p}}\right) = \lim_{b\to\infty}\left(\frac{1}{1-p}\left(\frac{1}{b^{1-p}} - 1\right)\right)$$

(e) $\displaystyle\int_1^\infty \frac{dx}{x^p}$, $p > 1$ (so $p - 1 > 0$)
$$= \lim_{b\to\infty}\left(\frac{1}{1-p}\cdot\frac{1}{b^{p-1}} - \frac{1}{1-p}\right) = \frac{1}{1-p}\cdot 0 - \frac{1}{1-p} = \frac{1}{p-1}$$

$\displaystyle\int_1^\infty \frac{dx}{x^p}$, $p < 1$ (so $p - 1 < 0$)
$$= \lim_{b\to\infty}\left(\frac{1}{1-p}\cdot\frac{1}{b^{p-1}} - \frac{1}{1-p}\right) = \lim_{b\to\infty}\left(\frac{1}{1-p}\cdot b^{1-p} - \frac{1}{1-p}\right) = \infty$$

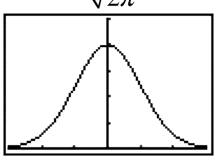
(f) The integral converges for $p > 1$ and diverges for $p \le 1$.

55. **(a)** $A(x) = \dfrac{\pi}{4} e^{2x}$

(b) $V = \int_{-\infty}^{\ln 2} A(x)\,dx = \int_{-\infty}^{\ln 2} \dfrac{\pi}{4} e^{2x}\,dx$

(c)
$$
\begin{aligned}
V &= \int_{-\infty}^{\ln 2} \frac{\pi}{4} e^{2x}\,dx \\
&= \lim_{b\to-\infty} \int_{b}^{\ln 2} \frac{\pi}{4} e^{2x}\,dx \\
&= \lim_{b\to-\infty} \left(\frac{\pi e^{2x}}{8}\right)\Bigg|_{b}^{\ln 2} \\
&= \frac{\pi 4}{8} - 0 \\
&= \frac{\pi}{2}
\end{aligned}
$$

56. **(a)** $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2}$



[–3, 3] by [0, 0.5]

f is increasing on $(-\infty, 0]$, f is decreasing on $[0, \infty)$. f has a local maximum at $(0, f(0) = \left(0, \dfrac{1}{\sqrt{2\pi}}\right)$

(b)
$$\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ x, -1, 1\right) \approx 0.683$$
$$\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ x, -2, 2\right) \approx 0.954$$
$$\text{NINT}\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2},\ x, -3, 3\right) \approx 0.997$$

(c) Part (b) suggests that as b increases, the integral approaches 1. We can make $\int_{-b}^{b} f(x)\,dx$ as close to 1 as we want by choosing $b > 1$ large enough. Also we can make $\int_{-b}^{\infty} f(x)\,dx$ and $\int_{\infty}^{-b} f(x)\,dx$ as small as we want by choosing b large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$). Thus, for $b > 1$, $\int_{b}^{\infty} f(x)\,dx < \int_{b}^{\infty} e^{-x/2}\,dx$.

$$
\begin{aligned}
\int_{b}^{\infty} e^{-x/2}\,dx &= \lim_{c\to\infty} \int_{b}^{c} e^{-x/2}\,dx \\
&= \lim_{c\to\infty} [-2e^{-x/2}]_{b}^{c} \\
&= \lim_{c\to\infty} [-2e^{-c/2} + 2e^{-b/2}] \\
&= 2e^{-b/2}
\end{aligned}
$$

As $b \to \infty$, $2e^{-b/2} \to 0$, so for large enough b, $\int_{b}^{\infty} f(x)\,dx$ is as small as we want. Likewise, for large enough b, $\int_{-\infty}^{-b} f(x)\,dx$ is as small as we want.

57. **(a)** For $x \ge 6$, $x^2 \ge 6x$, so $e^{-x^2} \le e^{-6x}$.

$$
\begin{aligned}
\int_{6}^{\infty} e^{-x^2}\,dx &\le \int_{6}^{\infty} e^{-6x}\,dx \\
&= \lim_{b\to\infty} \int_{6}^{b} e^{-6x}\,dx \\
&= \lim_{b\to\infty} \left[-\frac{1}{6} e^{-6x}\right]_{6}^{b} \\
&= \lim_{b\to\infty} \left(-\frac{1}{6} e^{-6b} + \frac{1}{6} e^{-36}\right) \\
&= \frac{1}{6} e^{-36} \\
&< 4\times 10^{-17}
\end{aligned}
$$

(b)
$$
\begin{aligned}
\int_{1}^{\infty} e^{-x^2}\,dx &= \int_{1}^{6} e^{-x^2}\,dx + \int_{6}^{\infty} e^{-x^2}\,dx \\
&\le \int_{1}^{6} e^{-x^2}\,dx + 4\times 10^{-17}
\end{aligned}
$$

Thus, from part (a) we have shown that the error is bounded by 4×10^{-17}.

(c)
$$
\begin{aligned}
\int_{1}^{\infty} e^{-x^2}\,dx &\approx \text{NINT}(e^{-x^2},\ x, 1.6) \\
&\approx 0.1394027926
\end{aligned}
$$

(This agrees with Figure 8.16.)

(d)
$$
\begin{aligned}
\int_{0}^{\infty} e^{-x^2}\,dx &= \int_{0}^{3} e^{x^2}\,dx + \int_{3}^{\infty} e^{x^2}\,dx \\
&\le \int_{0}^{3} e^{-x^2}\,dx + \int_{3}^{\infty} e^{-3x}\,dx
\end{aligned}
$$

since $x^2 \ge 3x$ for $x > 3$.

$$\int_3^\infty e^{-3x}\,dx = \lim_{b\to\infty}\int_3^b e^{-3x}\,dx$$
$$= \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}+\frac{1}{3}e^{-9}\right)$$
$$= \frac{1}{3}e^{-9} \approx 0.000041 < 0.000042$$

58. Suppose $0 \le f(x) \le g(x)$ for all $x \ge a$.
From the properties of integrals, for any $b > a$,
$\int_a^b f(x) \le \int_a^b g(x)\,dx.$
If the improper integral of g converges, then taking the limit in the above inequality as $b \to \infty$ shows that the improper integral of f is bounded above by the improper integral of g. Therefore, the improper integral of f must be finite and it converges. If the improper integral of f diverges, it must grow to infinity. So taking the limit in the above inequality as $b \to \infty$ shows that the improper integral of g must also diverge.

59. **(a)** For $n = 0$:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}[-e^{-x}]_0^b$$
$$= \lim_{b\to\infty}[-e^{-b}+1]$$
$$= 1$$

For $n = 1$:

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \qquad v = -e^{-x}$

$$\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left([-xe^{-x}]_0^b + \int_0^b e^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b}{e^b}\right) + \lim_{b\to\infty}\int_0^b e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{1}{e^b}\right)+1$$
$$= 1$$

For $n = 2$;

$u = x^2 \qquad dv = e^{-x}\,dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$\int_0^\infty x^2 e^{-x}\,dx$$
$$= \lim_{b\to\infty}\int_0^b x^2 e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left([-x^2e^{-x}]_0^b + \int_0^b 2xe^{-x}\,dx\right)$$
$$= \lim_{b\to\infty}\left(-\frac{b^2}{e^b}\right) + 2\lim_{b\to\infty}\int_0^b xe^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right)+2(1)$$
$$= \lim_{b\to\infty}\left(-\frac{2b}{e^b}\right)+2$$
$$= 2$$

(b) Evaluate $\int x^n e^{-x}\,dx$ using integration by parts

$u = x^n \qquad dv = e^{-x}\,dx$

$du = nx^{n-1} \qquad v = -e^{-x}$

$$\int x^n e^{-x}\,dx = -x^n e^{-x} + \int nx^{n-1}e^{-x}\,dx$$
$$f(n+1)$$
$$= \int_0^\infty x^n e^{-x}\,dx$$
$$= \lim_{b\to\infty}[-x^n e^{-x}]_0^b + \int_0^\infty nx^{n-1}e^{-x}\,dx$$
$$= \lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) + n\int_0^\infty x^{n-1}e^{-x}\,dx$$
$$= nf(n)$$

(Note: apply L'Hôpital's Rule n times to show that $\lim_{b\to\infty}\left(-\frac{b^n}{e^b}\right) = 0.$)

(c) Since $f(n+1) = nf(n)$,
$f(n+1) = n(n-1)\cdots f(1) = n!$; thus
$\int_0^\infty x^n e^{-x}\,dx$ converges for all integers $n \ge 0$.

60. **(a)** On a grapher, plot $\text{NINT}\left(\frac{\sin x}{x}, x, 0, x\right)$ or create a table of values. For large values of x, $f(x)$ appears to approach approximately 1.57.

(b) The integral is an infinite sum of finite areas that *alternate in sign, decrease in absolute value*, and tend to *a limit of zero.* The partial sums must therefore close in on some limit. (Figure 10.18 in Section 10.5 shows a diagram of what this convergence looks like.)

61. **(a)** $$\int_{-\infty}^{1} \frac{dx}{1+x^2} = \lim_{b\to-\infty} \int_b^1 \frac{dx}{1+x^2} = \lim_{b\to-\infty} [\tan^{-1} x]_b^1 = \lim_{b\to-\infty} (\tan^{-1} 1 - \tan^{-1} b) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$$

$$\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{x\to\infty} \int_1^b \frac{dx}{1+x^2} = \lim_{b\to\infty} [\tan^{-1} x]_1^b = \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{3\pi}{4} + \frac{\pi}{4} = \pi$$

(b) $$\int_{-\infty}^{c} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^c f(x)\,dx$$

$$\int_c^{\infty} f(x)\,dx = \int_c^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx$$

Thus, $\int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^c f(x)\,dx + \int_c^0 f(x)\,dx + \int_0^{\infty} f(x)\,dx$
$= \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx,$

because $\int_0^c f(x)\,dx + \int_c^0 f(x)\,dx = \int_0^c f(x)\,dx - \int_0^c f(x)\,dx = 0.$

Quick Quiz Section 9.3 and 9.4

1. E; $\lim_{x\to\infty} \frac{0.1x^3}{x^2} = \lim_{x\to\infty} \frac{0.3x^2}{2x} = \lim_{x\to\infty} \frac{0.6x}{2} = \infty.$

2. C; since $\int_1^{\infty} \frac{dx}{x^p} = \frac{1}{p-1}$ for $p > 1$, and $p + 1 > 1$ when $p > 0$.

3. B; $\int_0^1 \frac{dx}{x^{p+1}} = C$ when $p < 0$.

4. **(a)** $\int_1^{\infty} \frac{2\ln(x)}{x^2}\,dx$

(b) $\lim_{b\to\infty} \int_1^b \frac{2\ln x}{x^2}\,dx$

(c) Note that $\int \frac{2\ln x}{x^2}\,dx$ can be found by parts. Let $u = 2\ln x$ and $dv = x^{-2}\,dx$.

Then $\int \frac{2\ln x}{x^2}\,dx$

$$= 2\ln x(-x^{-1}) - \int (-x^{-1})\,2x^{-1}\,dx$$
$$= -\frac{2\ln x}{x} + \int \frac{2}{x^2}\,dx$$
$$= -\frac{2\ln x}{x} - \frac{2}{x} + C$$

$$\text{Area} = \lim_{b\to\infty} \int_1^b \frac{2\ln x}{x^2}\,dx$$
$$= \lim_{b\to\infty} \left[-\frac{2\ln x}{x} - \frac{2}{x}\right]_1^b$$
$$= \lim_{b\to\infty} \left(-\frac{2\ln b}{b} - \frac{2}{b} + 0 + 2\right)$$
$$= 2$$

Chapter 9 Review Exercises (pp. 474–475)

1. $-\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}, \frac{5}{7}$

$a_{40} = (-1)^{40}\frac{40+1}{40+3} = \frac{41}{43}$

2. $-3, -6, -12, -24$

$a_{40} = -3(2^{39})$

3. (a) $\frac{1}{2} - (-1) = \frac{3}{2}$

(b) $-1 + 9\left(\frac{3}{2}\right) = \frac{25}{2}$

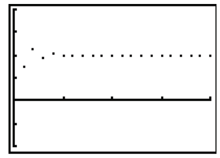
(c) $a_n = -1 + (n-1)\frac{3}{2} = \frac{3n-5}{2}$

4. (a) $\frac{-2}{\frac{1}{2}} = -4$

(b) $\frac{1}{2}(-4)^6 = 2048$

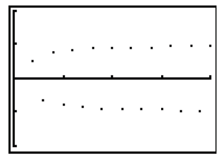
(c) $a_n = (-1)^{n-1}\left(\frac{1}{2}\right)4^{n-1}$
$= (-1)^{n-1}(2^{2n-3})$

5.



[0, 20] by [–2, 4]

6.



[0, 20] by [–2, 2]

7. $a_n = \lim_{n\to\infty} \frac{3n^2-1}{2n^2+1}$
$= \lim_{n\to\infty} \frac{6n}{4n}$
$= \lim_{n\to\infty} \frac{3}{2}$
$= \frac{3}{2}$, it converges.

8. The sequence diverges, since

$\lim_{n\to\infty} (-1)^n \frac{3n-1}{n+2} = 3$ for n even, and

$\lim_{n\to\infty} (-1)^n \frac{3n-1}{n+2} = -3$ for n odd.

9. $\lim_{t\to 0} \frac{t - \ln(1+2t)}{t^2} = \lim_{t\to 0} \frac{1 - \frac{2}{1+2t}}{2t} = \infty$ for $t \to 0^-$ and $-\infty$ for $t \to 0^+$

The limit does not exist.

10. $\lim_{t\to 0} \frac{\tan 3t}{\tan 5t} = \lim_{t\to 0} \frac{3\sec^2 3t}{5\sec^2 5t} = \frac{3}{5}$

11. $\lim_{x\to 0} \frac{x\sin x}{1-\cos x} = \lim_{x\to 0} \frac{x\cos x + \sin x}{\sin x}$
$= \lim_{x\to 0} \frac{-x\sin x + \cos x + \cos x}{\cos x}$
$= 2$

12. The limit leads to the indeterminate form 1^∞.

$$f(x) = x^{1/(1-x)}$$

$$\ln f(x) = \frac{\ln x}{1-x}$$

$$\lim_{x\to 1} \frac{\ln x}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

$$\lim_{x\to 1} x^{1/(1-x)} = \lim_{x\to 1} e^{\ln f(x)} = e^{-1} = \frac{1}{e}$$

13. The limit leads to the indeterminate form ∞^0.

$$f(x) = x^{1/x}$$

$$\ln f(x) = \frac{\ln x}{x}$$

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1$$

14. The limit leads to the indeterminate form 1^∞.

$$f(x) = \left(1+\frac{3}{x}\right)^x$$

$$\ln f(x) = x\ln\left(1+\frac{3}{x}\right) = \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}}$$

$$\lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{\frac{-3}{x^2}}{1+\frac{3}{x}}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{3x}{x+3} = 3$$

$$\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = \lim_{x\to\infty} e^{\ln f(x)} = e^3$$

15. $\lim\limits_{r\to\infty} \dfrac{\cos r}{\ln r} = 0$ since $|\cos r| \le 1$ and $\ln r \to \infty$ as $r \to \infty$.

16.

$$\lim_{\theta\to\pi/2} \left(\theta - \frac{\pi}{2}\right)\sec\theta = \lim_{\theta\to\pi/2} \frac{\theta-\frac{\pi}{2}}{\cos\theta} = \lim_{\theta\to\pi/2} \frac{1}{-\sin\theta} = -1$$

17.

$$\lim_{x\to 1}\left(\frac{1}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1}\left[\frac{\ln x - x + 1}{(x-1)\ln x}\right] = \lim_{x\to 1} \frac{\frac{1}{x}-1}{\frac{x-1}{x}+\ln x} = \lim_{x\to 1} \frac{1-x}{x-1+x\ln x} = \lim_{x\to 1} \frac{-1}{2+\ln x} = -\frac{1}{2}$$

18. The limit leads to the indeterminate form ∞^0.

$$f(x) = \left(1+\frac{1}{x}\right)^x$$

$$\ln f(x) = x\ln\left(1+\frac{1}{x}\right) = \frac{\ln[\left[1+\frac{1}{x}\right]]}{\frac{1}{x}}$$

$$\lim_{x\to 0^+} \frac{\ln\left[1+\frac{1}{x}\right]}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x}{x+1} = 0$$

$$\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1$$

19. The limit leads to the indeterminate form 0^0.

$$f(\theta) = (\tan\theta)^\theta$$

$$\ln f(\theta) = \theta\ln(\tan\theta) = \frac{\ln(\tan\theta)}{\frac{1}{\theta}}$$

$$\lim_{x\to 0^+} \frac{\ln(\tan\theta)}{1/\theta} = \lim_{x\to 0^+} \frac{\frac{\sec^2\theta}{\tan\theta}}{-\frac{1}{\theta^2}} = \lim_{x\to 0^+} -\frac{\theta^2}{\sin\theta\cos\theta} = \lim_{x\to 0^+} \frac{-2\theta}{-\sin^2\theta + \cos^2\theta} = 0$$

$$\lim_{x\to 0^+} (\tan\theta)^\theta = \lim_{x\to 0^+} e^{\ln f(\theta)} = e^0 = 1$$

20.

$$\lim_{\theta\to\infty} \theta^2 \sin\left(\frac{1}{\theta}\right) = \lim_{t\to 0^+} \frac{\sin t}{t^2} = \lim_{t\to 0^+} \frac{\cos t}{2t} = \infty$$

21. $\lim_{x\to\infty} \frac{x^3-3x^2+1}{2x^2+x-3} = \lim_{x\to\infty} \frac{3x^2-6x}{4x+1}$
$= \lim_{x\to\infty} \frac{6x-6}{4}$
$= \infty$

22. $\lim_{x\to\infty} \frac{3x^2-x+1}{x^4-x^3+2} = \lim_{x\to\infty} \frac{6x-1}{4x^3-3x^2}$
$= \lim_{x\to\infty} \frac{6}{12x^2-6x}$
$= 0$

23. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{5x} = \frac{1}{5}$

f grows at the same rate as *g*.

24. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\log_2 x}{\log_3 x}$
$= \lim_{x\to\infty} \frac{\frac{\ln x}{\ln 2}}{\frac{\ln x}{\ln 3}}$
$= \frac{\ln 3}{\ln 2}$

f grows at the same rate as *g*.

25. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\frac{x^2+1}{x}}$
$= \lim_{x\to\infty} \frac{x^2}{x^2+1}$
$= \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$
$= 1$

f grows at the same rate as *g*.

26. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\frac{x}{100}}{xe^{-x}} = \lim_{x\to\infty} \frac{e^x}{100} = \infty$

f grows faster than *g*.

27. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\tan^{-1} x} = \infty$ since

$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$ and $\lim_{x\to\infty} x = \infty$

f grows faster than *g*.

28. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{\csc^{-1} x}{\frac{1}{x}}$
$= \lim_{x\to\infty} \frac{-\frac{1}{x\sqrt{x^2-1}}}{-\frac{1}{x^2}}$
$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}}$
$= \lim_{x\to\infty} \sqrt{\frac{x^2}{x^2-1}}$
$= \lim_{x\to\infty} \sqrt{\frac{1}{1-\frac{1}{x^2}}}$
$= 1$

f grows at the same rate as *g*.

29. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\ln x}}{x^{\log_2 x}}$
$= \lim_{x\to\infty} x^{\ln x - \log_2 x}$
$= \lim_{x\to\infty} x^{\ln x - (\ln x)/\ln 2}$
$= \lim_{x\to\infty} x^{(\ln x)(1-1/\ln 2)}$
$= \lim_{x\to\infty} \left(\frac{1}{x}\right)^{(\ln x)(1/\ln 2 - 1)}$
$= 0$

Note that $1-\frac{1}{\ln 2} < 0$ since $\ln 2 < 1$.

f grows slower than *g*.

30. $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{3^{-x}}{2^{-x}}$
$= \lim_{x\to\infty} \frac{2^x}{3^x}$
$= \lim_{x\to\infty} \left(\frac{2}{3}\right)^x$
$= 0$ since $\frac{2}{3} < 1$.

f grows slower than *g*.

31. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\ln 2x}{\ln x^2}=\lim_{x\to\infty}\frac{\ln x+\ln 2}{2\ln x}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{2}{x}}=\frac{1}{2}$$

f grows at the same rate as g.

32. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{10x^3+2x^2}{e^x}=\lim_{x\to\infty}\frac{30x^2+4x}{e^x}=\lim_{x\to\infty}\frac{60x+4}{e^x}=\lim_{x\to\infty}\frac{60}{e^x}=0$$

f grows slower than g.

33. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{1+\left(\frac{1}{x}\right)^2}(-x^{-2})}{-x^{-2}}=\lim_{x\to\infty}\frac{1}{1+\left(\frac{1}{x}\right)^2}=1$$

f grows at the same rate as g.

34. $$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{x\to\infty}\frac{\sin^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}(-x^{-2})}{-2x^{-3}}=\lim_{x\to\infty}\frac{x}{2\sqrt{1-\left(\frac{1}{x}\right)^2}}=\infty$$

f grows faster than g.

35. **(a)** $$\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{2^{\sin x}-1}{e^x-1}=\lim_{x\to 0}\frac{(\ln 2)(\cos x)2^{\sin x}}{e^x}=\ln 2$$

(b) Define $f(0)=\ln 2$.

36. **(a)** $$\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}x\ln x=\lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0^+}\frac{\frac{1}{x}}{-\left(\frac{1}{x^2}\right)}=\lim_{x\to 0^+}(-x)=0$$

(b) Define $f(0)=0$.

37. $$\int_1^\infty\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^{3/2}}=\lim_{b\to\infty}\left(\frac{-2}{\sqrt{x}}\right)\Bigg|_1^b=2$$

38. $$\int_1^\infty\frac{dx}{x^2+7x+12}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^2+7x+12}=\lim_{b\to\infty}\left[\int_1^b\frac{dx}{(x+3)}-\int_1^b\frac{dx}{(x+4)}\right]=\lim_{b\to\infty}\left(-\ln\left(\frac{|x+4|}{|x+3|}\right)\right)\Bigg|_1^b=\ln\frac{5}{4}$$

39. $$\int_{-\infty}^{-1}\frac{3dx}{3x-x^2}=\lim_{b\to-\infty}\int_b^{-1}\frac{3dx}{3x-x^2}=\lim_{b\to-\infty}\left(-\ln\left(\frac{|x-3|}{|x|}\right)\right)\Bigg|_b^{-1}=-2\ln(2)$$

40. $$\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3}\int_0^3\frac{dx}{\sqrt{9-x^2}}=\lim_{b\to 3}\left(\sin^{-1}\frac{x}{3}\right)\Bigg|_0^b=\frac{\pi}{2}$$

41. $\int_0^1 \ln(x)\,dx = \lim_{b\to 0}\int_b^1 \ln(x)\,dx$

$= \lim_{b\to 0}(x\ln x - x)\Big|_b^1$

$= -1$

42. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \lim_{b\to 0^-}\int_{-1}^b \frac{dy}{y^{2/3}} + \lim_{b\to 0^+}\int_b^1 \frac{dy}{y^{2/3}}$

$= \lim_{b\to 0^-}\left(\frac{-2}{\sqrt{y}}\right)\Big|_{-1}^b + \lim_{b\to 0^+}\left(\frac{-2}{\sqrt{y}}\right)\Big|_b^1$

$= 6$

43. $\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

$= \lim_{b\to -1^-}\int_{-2}^b \frac{d\theta}{(\theta+1)^{3/5}} + \lim_{b\to -1^+}\int_b^0 \frac{d\theta}{(\theta+1)^{3/5}}$

$= \lim_{b\to -1^{-1}}\left(\frac{5}{2}(x+1)^{\frac{2}{5}}\right)\Bigg|_{-2}^b + \lim_{b\to -1^+}\left(\frac{5}{2}(x+1)^{\frac{2}{5}}\right)\Bigg|_b^0$

$= 0$

44. $\int_3^\infty \frac{2dx}{x^2-2x} = \lim_{b\to\infty}\int_3^b \frac{2dx}{x^2-2x}$

$= \lim_{b\to\infty}\left(\ln\left(\frac{|x-2|}{|x|}\right)\right)\Bigg|_3^b$

$= \ln 3$

45. $\int_0^\infty x^2 e^{-x}dx = \lim_{b\to\infty}\int_0^b x^2e^{-x}dx$

$= \lim_{b\to\infty}((-x^2-2x-2)e^{-x})\Big|_0^b$

$= 2$

46. $\int_{-\infty}^0 xe^{3x}dx = \lim_{b\to-\infty}\int_b^0 xe^{3x}dx$

$= \lim_{b\to-\infty}\left(\left(\frac{x}{3}-\frac{1}{9}\right)e^{3x}\right)\Bigg|_b^0$

$= -\frac{1}{9}$

47. $\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

$= \lim_{b\to-\infty}\int_b^0 \frac{dx}{e^x+e^{-x}} + \lim_{b\to\infty}\int_0^b \frac{dx}{e^x+e^{-x}}$

$= \lim_{b\to-\infty}(\tan^{-1}e^x)\Big|_b^0 + \lim_{b\to\infty}(\tan^{-1}(e^x))\Big|_0^b$

$= \frac{\pi}{2}$

48. $\int_{-\infty}^\infty \frac{4dx}{x^2+16}$

$= \lim_{b\to-\infty}\int_b^0 \frac{4dx}{x^2+16} + \lim_{b\to\infty}\int_0^b \frac{4dx}{x^2+16}$

$= \lim_{b\to-\infty}\left(\tan^{-1}\left(\frac{x}{4}\right)\right)\Bigg|_b^0 + \lim_{b\to-\infty}\left(\tan^{-1}\left(\frac{x}{4}\right)\right)\Bigg|_0^b$

$= \pi$

49. $\int_1^\infty \frac{\ln z}{z}dz = \int_1^e \frac{\ln z}{z}dz + \int_e^\infty \frac{\ln z}{z}dz$

$> \int_1^e \frac{\ln z}{z}dz + \int_e^\infty \frac{1}{z}dz$

$= \infty$ [See Example 2 in Section 9.4.]

So $\int_1^\infty \frac{\ln z}{z}dz$ diverges.

50. $0 \le \int_1^\infty \frac{e^{-t}}{\sqrt{t}}dt \le \int_1^\infty e^{-t}\,dt$

$\int_1^\infty e^{-t}\,dt = \lim_{b\to\infty}\int_1^b e^{-t}\,dt$

$= \lim_{b\to\infty}(-e^{-t})\Big|_1^b$

$= -0 + e$

$= e^{-1}$

$\int_1^\infty \frac{e^{-t}}{\sqrt{t}}dt$ converges

51. **(a)** $\left(\frac{-\frac{3}{8}}{3}\right)^{1/3} = \frac{1}{2}$

$-3\left(\frac{1}{\frac{1}{2}}\right) = -6$

(b) $\frac{1}{2}$

(c) $a_n = -6\left(\frac{1}{2}\right)^{n-1}$

$a_n = -3(2^{2-n})$

52. **(a)** $\left(\frac{5.5-11.5}{4}\right) = -1.5$

$11.5-(-1.5) = 13$

(b) -1.5

(c) $a_n = 13+(n-1)(-1.5)$

$a_n = -1.5n+14.5$

53. **(a)** $\int_{-\infty}^{\infty} e^{-2|x|}\,dx$

$= \lim_{b\to-\infty}\int_b^0 e^{2x}\,dx + \lim_{b\to\infty}\int_0^b e^{-2x}\,dx$

(b) $\lim_{b\to-\infty}\int_b^0 e^{2x}\,dx + \lim_{b\to\infty}\int_0^b e^{-2x}\,dx$

$= \lim_{b\to-\infty}\left(\frac{e^{2x}}{2}\right)\Bigg|_b^0 - \lim_{b\to\infty}\left(\frac{e^{-2x}}{2}\right)\Bigg|_0^b$

$= 0+\frac{1}{2}+0+\frac{1}{2}$

$= 1$

54. For $x \ge 0$, $y \ge 0$ on $(0, 1]$.

$\text{Volume} = \int_0^1 \pi(-\ln x)^2\,dx$

$= \pi\int_0^1 (\ln x)^2\,dx$

$= \pi \lim_{b\to 0^+}\int_b^1 (\ln x)^2\,dx$

Evaluate $\int (\ln x)^2\,dx$ by integration by parts.

$u = (\ln x)^2 \qquad dv = dx$

$du = \frac{2(\ln x)}{x}dx \qquad v = x$

$\int (\ln x)^2\,dx = x(\ln x)^2 - \int 2\ln x\,dx$

Evaluate $\int 2\ln x\,dx$ using integration by parts.

$u = 2\ln x \qquad dv = dx$

$u = 2\ln x \qquad du = \frac{2}{x}dx \quad v = x$

$\int 2\ln x\,dx = 2x\ln x - \int 2\,dx = 2x\ln x - 2x + C$

$\int (\ln x)^2\,dx = x(\ln x)^2 - 2x\ln x + 2x + C$

$\text{Area} = \pi \lim_{b\to 0^+}[x(\ln x)^2 - 2x\ln x + 2x]_b^1$

$= \pi \lim_{b\to 0^+}[2 - b(\ln b)^2 + 2b\ln b - 2b]$

$= 2\pi - \lim_{b\to 0^+}\frac{\pi(\ln b)^2}{\frac{1}{b}} + 2\lim_{b\to 0^+}\frac{\pi\ln b}{\frac{1}{b}}$

$= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)\left(\frac{1}{b}\right)}{-\frac{1}{b^2}} + 2\lim_{b\to 0^+}\frac{\frac{\pi}{b}}{-\frac{1}{b^2}}$

$= 2\pi - \lim_{b\to 0^+}\frac{2\pi(\ln b)}{-\frac{1}{b}} + 2\lim_{b\to 0^+}(-\pi b)$

$= 2\pi - \lim_{b\to 0^+}\frac{\frac{2\pi}{b}}{\frac{1}{b^2}} - \lim_{b\to 0^+} 2\pi b$

$= 2\pi$

55. For $x \ge 0$, $y \ge 0$ on $[0, \infty)$.

$\text{Area} = \int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x}\,dx$

Evaluate $\int xe^{-x}\,dx$ by using integration by parts.

$u = x \qquad dv = e^{-x}\,dx$

$du = dx \quad v = -e^{-x}$

$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$

$\text{Area} = \lim_{b\to\infty}[-xe^{-x} - e^{-x}]_0^b$

$= \lim_{b\to\infty}[-be^{-b} - e^{-b} + 1]$

$= -\lim_{b\to\infty}\frac{b}{e^b} + 1$

$= -\lim_{b\to\infty}\frac{1}{e^b} + 1$

$= 1$

56. **(a)** $\int_0^\infty xe^{-x/2}\,dx$

(b) $\lim_{b\to\infty}\int_0^b xe^{-x/2}\,dx$

(c) Note that $\int xe^{-x/2}\,dx$ can be found by parts:

$$\int xe^{-x/2}\,dx = x(-2e^{-x/2}) - \int(-2e^{-x/2})\,dx$$
$$= -2xe^{-x/2} - 4e^{-x/2} + C.$$

$$\text{Area} = \lim_{b\to\infty}\int_0^b xe^{-x/2}\,dx$$
$$= \lim_{b\to\infty}[-2xe^{-x/2} - 4e^{-x/2}]_0^b$$
$$= \lim_{b\to\infty}(-2be^{-b/2} - 4e^{-b/2}\ 0 + 4)$$
$$= 4.$$

57. **(a)** $\int_0^\infty \pi x^2\,dy = \pi\int_0^\infty \frac{dy}{(y+1)^2}$

(b) $\lim_{b\to\infty}\pi\int_0^b \frac{dy}{(y+1)^2}$

(c) $$\text{Volume} = \lim_{b\to\infty}\pi\int_0^b \frac{dy}{(y+1)^2}$$
$$= \lim_{b\to\infty}\pi[-(y+1)^{-1}]_0^b$$
$$= \lim_{b\to\infty}\pi\left(-\frac{1}{b+1}+1\right)$$
$$= \pi$$

58. Note that $\int xe^{-x}\,dx$ can be found by parts:

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int(-e^{-x})dx$$
$$= -xe^{-x} - e^{-x} + C$$

So

$$\int_0^\infty xe^{-x}\,dx = \lim_{k\to\infty}\int_0^k xe^{-x}\,dx$$
$$= \lim_{k\to\infty}[-xe^{-x} - e^{-x}]_0^k$$
$$= \lim_{k\to\infty}\left(-\frac{k}{e^k} - \frac{1}{e^k} + 1\right)$$

By L'Hopital's rule,

$\lim_{k\to\infty}\left(-\frac{k}{e^k}\right) = \lim_{k\to\infty}\left(-\frac{1}{e^k}\right) = 0.$ Therefore,

$$\int_0^\infty xe^{-x}\,dx = \lim_{k\to\infty}\left(-\frac{k}{e^k} - \frac{1}{e^k} + 1\right)$$
$$= 0 - 0 + 1$$
$$= 1$$

The integral converges to 1.